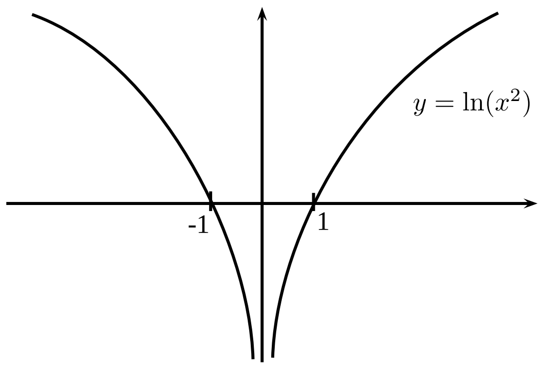
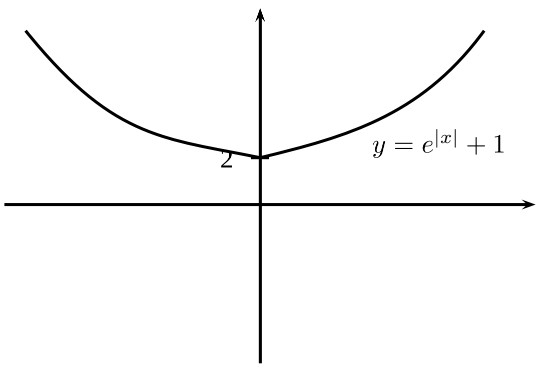
1. (a)



$R_f = \mathbb{R}$.

(b)



From the graph, $R_g = [2, \infty)$.
$D_f = \mathbb{R}\backslash\{0\}$.

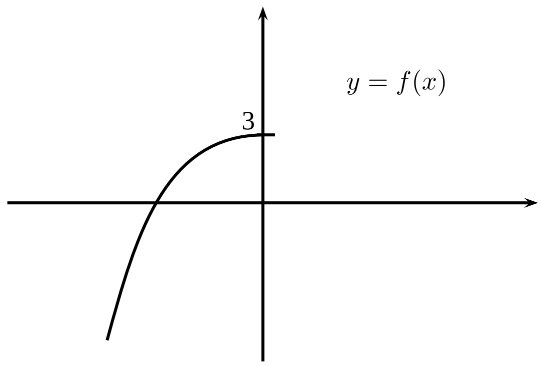
Since, $[2, \infty) \subseteq \mathbb{R}\backslash\{0\}$
$\Rightarrow R_g \subseteq D_f$
Therefore, fg exists.

(c)
Since the line $y = 1$ cuts the graph at two points, f is not $1-1$ and hence its inverse does not exist.

(d) $a = 0$.

$$\begin{aligned} y &= \ln(x^2) \\ e^y &= x^2 \\ x &= -\sqrt{e^y} \text{ or } \sqrt{e^y} \text{ (rej)} \\ f^{-1}(x) &= -e^{\frac{x}{2}}, x \in \mathbb{R} \end{aligned}$$

2. (a)

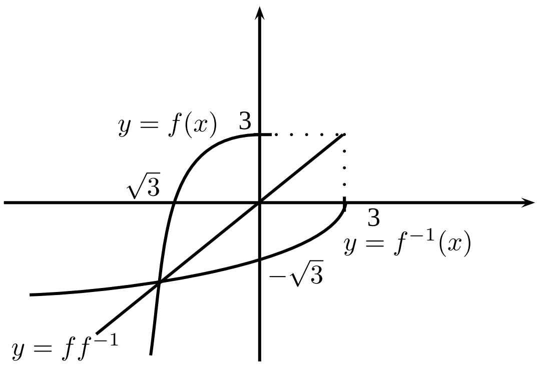


$R_f = (-\infty, 3]$

$$\begin{aligned} y &= -x^2 + 3 \\ y - 3 &= -x^2 \\ x^2 &= 3 - y \\ x &= -\sqrt{3-y} \text{ or } \sqrt{3-y} \text{ (rej as } x \leq 0) \end{aligned}$$

Therefore , $f^{-1}(x) = -\sqrt{3-x}, x \in (-\infty, 3]$

(b)



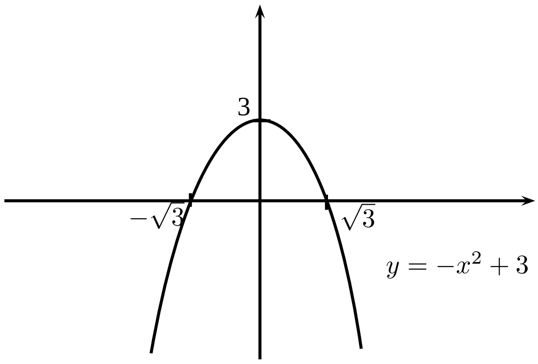
(Make sure your domain for ff^{-1} is correct!)

(c)

$$\begin{aligned} f(x) &= f^{-1}(x) \\ \Rightarrow f(x) &= x \\ -x^2 + 3 &= x \\ 0 &= x^2 + x - 3 \\ x &= 1.303 \text{(rej) or } -2.303 \end{aligned}$$

(d)
$R_f = (-\infty, 3]$
$D_h = (0, \infty)$
Since $R_f \not\subseteq D_h, hf$ cannot be formed.

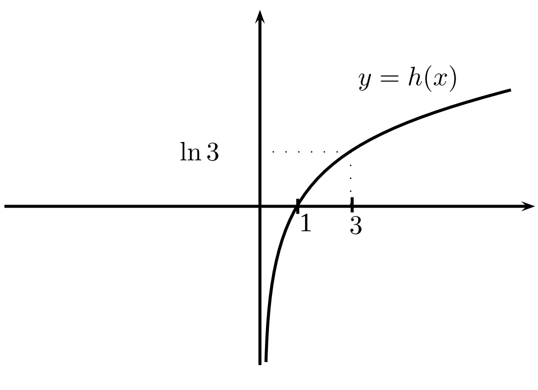
(e)



We want $R_g \subseteq D_h = (0, \infty)$.
In order for this to occur, maximal range of g, is, $R_g = (0, 3]$. This holds when domain $T = (-\sqrt{3}, \sqrt{3})$.

$h(g(x)) = h(-x^2 + 3) = \ln(-x^2 + 3)$
$D_{hg} = D_g = (-\sqrt{3}, \sqrt{3})$

$R_g = (0, 3]$



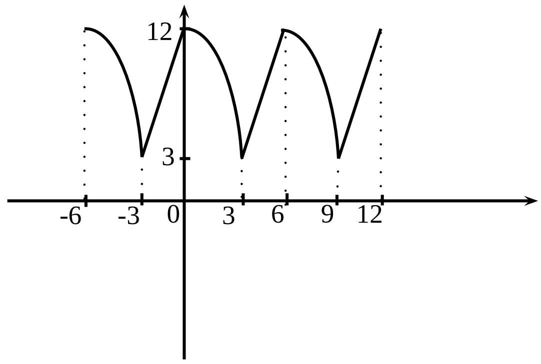
Using R_g as D_h, $R_{hg} = (-\infty, \ln 3]$

3. $h(x) = h(x - 6)$

$$\begin{aligned} h(16) &= h(16 - 6) \\ &= h(10) \\ &= h(10 - 6) \\ &= h(4) \\ &= 3(4) - 6 \\ &= 6 \end{aligned}$$

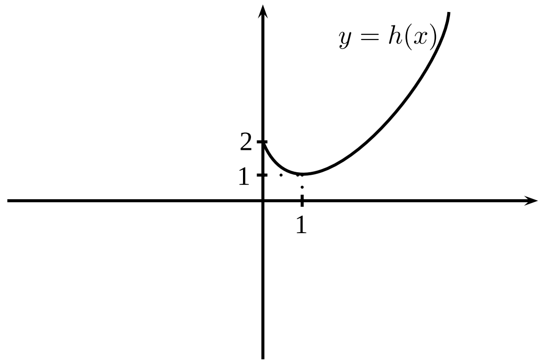
$$\begin{aligned} h(25) &= h(25 - 6) \\ &= h(19) \\ &= h(19 - 6) \\ &= h(13) \\ &= h(13 - 6) \\ &= h(7) \\ &= h(7 - 6) \\ &= h(1) \\ &= 12 - 1 \\ &= 11 \end{aligned}$$

Hence, $h(16) + h(25) = 6 + 11 = 17$

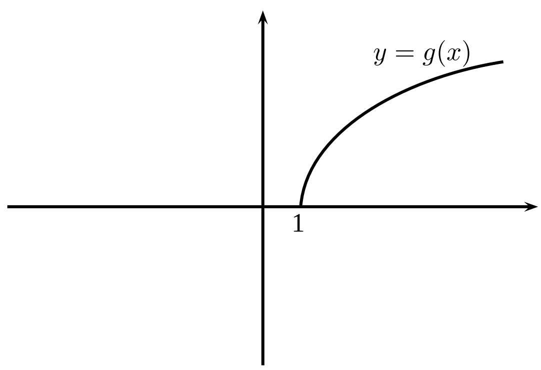


4. (i)

$$\begin{aligned} g(h(x)) &= g(x^2 - 2x + 2) \\ &= \ln(x^2 - 2x + 2), \quad x > 0 \end{aligned}$$

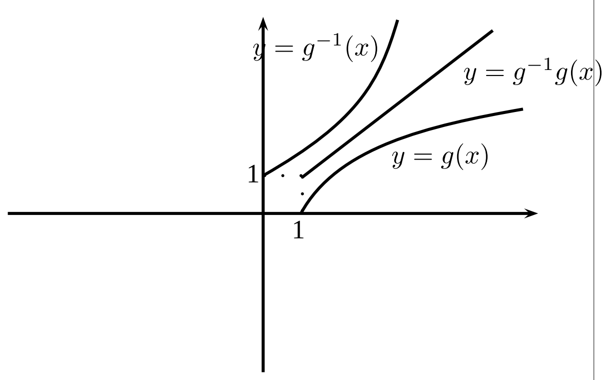


$R_h = [1, \infty)$
Use range h as domain g:



$\therefore R_{gh} = [0, \infty)$

(ii)

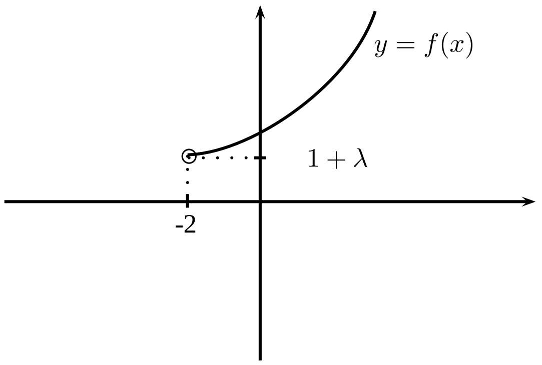


(iii)

$gg^{-1}(x) = x, \quad x \geq 0$
$g^{-1}g(x) = x, \quad x \geq 1$
$\therefore$ the two graphs intersect in the range $x \geq 1$.

5. (a)

$$\begin{aligned} f(x) &= x^2 + 4x + 5 + \lambda \\ &= (x+2)^2 - 2^2 + 5 + \lambda \\ &= (x+2)^2 + 1 + \lambda \end{aligned}$$



$\therefore R_f = (1 + \lambda, \infty)$

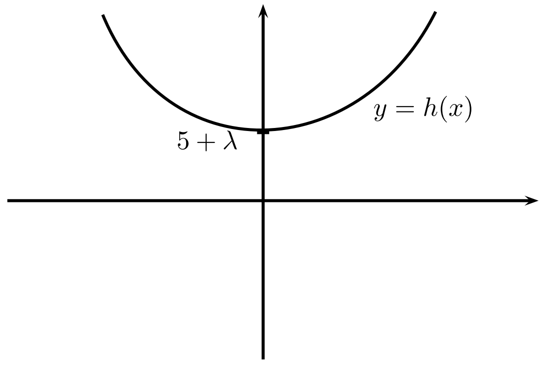
(b)

$$\begin{aligned} y &= (x+2)^2 + 1 + \lambda \\ y - 1 - \lambda &= (x+2)^2 \\ \pm\sqrt{y - 1 - \lambda} &= x + 2 \end{aligned}$$

$x = -2 + \sqrt{y - 1 - \lambda}$ or $x = -2 - \sqrt{y - 1 - \lambda}$ (rej)

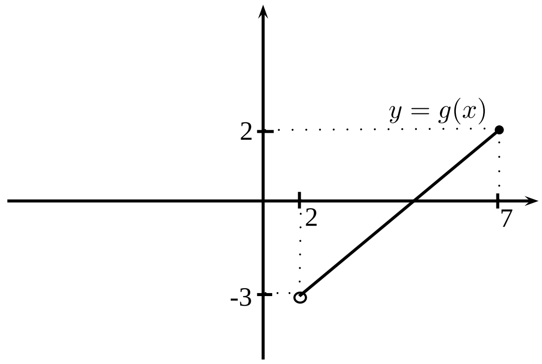
$\therefore f^{-1}(x) = -2 + \sqrt{x - 1 - \lambda}, \quad x \in (1 + \lambda, \infty)$

(c) Replace x by $|x|$.

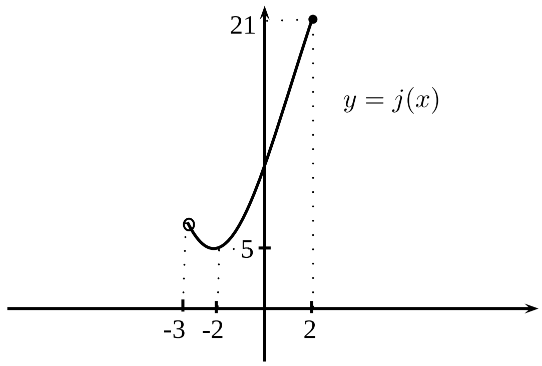


$R_h = [5 + \lambda, \infty)$

(d)

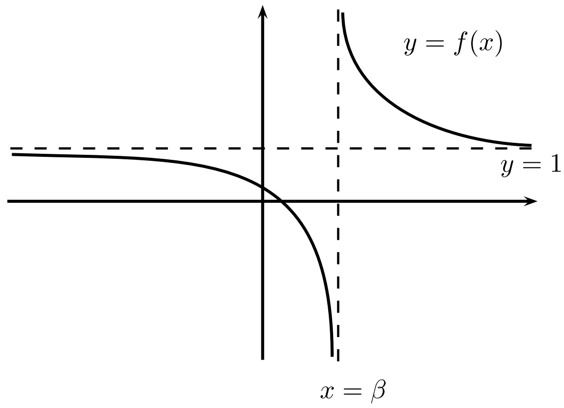


$R_g = (-3, 2]$



Using R_g as D_j, $R_{jg} = [5, 21]$

6. (a)
$f(x) = \frac{(x-\beta)+\beta-\alpha}{x-\beta} = 1 + \frac{\beta-\alpha}{x-\beta}$



From the graph, any horizontal line $y = k$ will cut the graph at most once. Hence f is $1-1$ and f^{-1} exists.

(b)
If $\beta = 1, \quad f(x) = \frac{x-\alpha}{x-1}$.

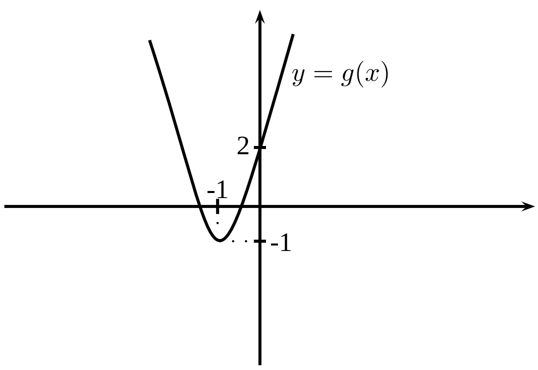
$$\begin{aligned} y &= \frac{x-\alpha}{x-1} \\ (x-1)y &= x-\alpha \\ xy - y &= x - \alpha \\ xy - x &= y - \alpha \\ x(y-1) &= y-\alpha \\ x &= \frac{y-\alpha}{y-1} \end{aligned}$$

$$\therefore f^{-1}(x) = \frac{x-\alpha}{x-1}, \quad D_{f^{-1}}(x) = R_f = \mathbb{R}\backslash\{1\}$$

Hence, $f^{-1}(x) = f(x)$

$$\begin{aligned} f^{-1}(x) &= f(x) \\ x &= f^2(x) \\ f^{2012}(x) &= (f^2)^{1006}(x) = x \\ \therefore f^{2013}(x) &= f(x) = \frac{x-\alpha}{x-1} \\ f^{2013}(\alpha) &= 0 \end{aligned}$$

(c)



From the graph, $R_g = [-1, \infty)$
We want $R_g \not\subseteq D_f$
$[-1, \infty) \not\subseteq \mathbb{R}\backslash\{\beta\}$

For the above to happen, β must belong to $[-1, \infty)$.

$\therefore \beta \in [-1, \infty)$

(d)
$f(g(x)) = f(3x^2 + 6x + 2) = \frac{3x^2+6x+2-\alpha}{3x^2+6x+2-\beta}$

$$\begin{aligned} fg(0) = \frac{2-\alpha}{2-\beta} &= 2 \\ 2 - \alpha &= 4 - 2\beta \\ \alpha &= 2 - 4 + 2\beta \\ &= -2 + 2\beta \end{aligned}$$

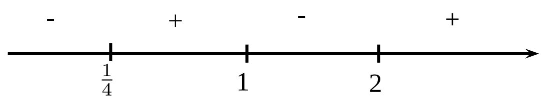
For fg to exist, from (c), $\beta < -1$
Hence,

$$\begin{aligned} \alpha &= -2 + 2\beta \\ &< -2 - 2 \\ &= -4 \end{aligned}$$

Inequalities

1. (a)

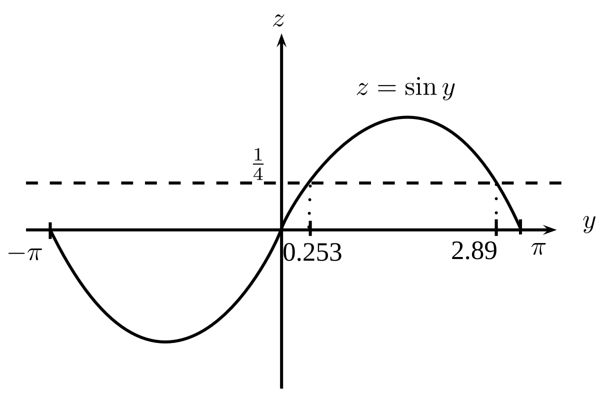
$$\begin{aligned}\frac{3}{1-x} &\leq 5-4x, \quad x \neq 1\\ \frac{3}{1-x}-5+4x &\leq 0\\ \frac{3}{1-x}+\frac{(-5+4x)(1-x)}{1-x} &\leq 0\\ \frac{3+(-5+5x+4x-4x^2)}{1-x} &\leq 0\\ \frac{-2+9x-4x^2}{1-x} &\leq 0\\ \frac{(-4x+1)(x-2)}{1-x} &\leq 0\end{aligned}$$



Since $x \neq 1$, $x \leq \frac{1}{4}, 1 < x \leq 2$

(b)
From $\frac{3}{1-x} \leq 5-4x$, replacing x by $\sin y$,
$\frac{3}{1-\sin y} \leq 5-4\sin y$

$\therefore \sin y \leq \frac{1}{4}, \quad 1 < \sin y \leq 2$ (N.A)



From the graph,
$-\pi \leq y \leq 0.253, \quad 2.89 \leq y \leq \pi$

2.

$$\begin{aligned}\frac{4x^2+4x+1}{x^2+x+1} &> 0\\ \frac{(2x+1)^2}{x^2+x+1} &> 0\\ \frac{(2x+1)^2}{(x+\frac{1}{2})^2-(\frac{1}{2})^2+1} &> 0\\ \frac{(2x+1)^2}{(x+\frac{1}{2})^2+\frac{3}{4}} &> 0\end{aligned}$$

Since $(x+\frac{1}{2})^2+\frac{3}{4}$ is always positive, we get,

$$(2x+1)^2 > 0$$

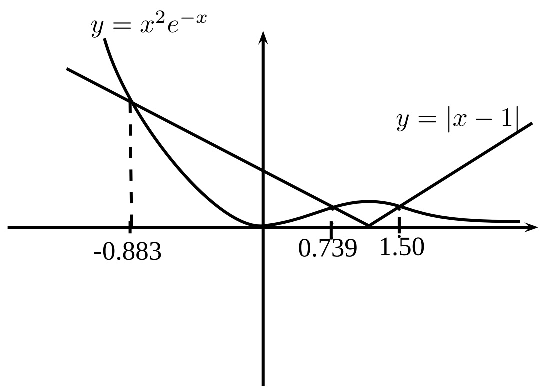
$\therefore x \in \mathbb{R}, x \neq -\frac{1}{2}$
(since the only value that makes this inequality false is when $x = -\frac{1}{2}$.)

From $\frac{4x^2+4x+1}{x^2+x+1} > 0$, replacing x by $\frac{1}{x}$,

$$\begin{aligned}\frac{\frac{4}{x^2}+\frac{4}{x}+1}{\frac{1}{x^2}+\frac{1}{x}+1} &> 0\\ \frac{\frac{4}{x^2}+\frac{4x}{x^2}+\frac{x^2}{x^2}}{\frac{1}{x^2}+\frac{x}{x^2}+\frac{x^2}{x^2}} &> 0\\ \frac{4+4x+x^2}{1+x+x^2} &> 0\end{aligned}$$

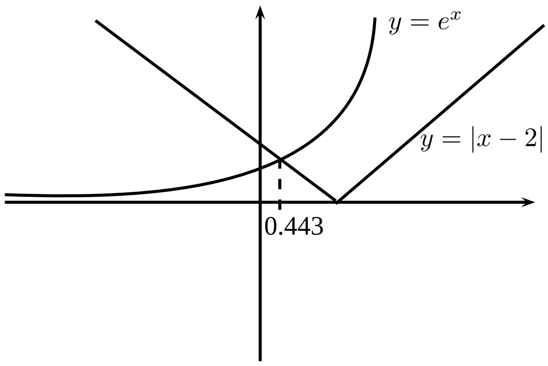
Replacing x by $\frac{1}{x}$ for the answer in the first part,
$\frac{1}{x} \in \mathbb{R}, \frac{1}{x} \neq \frac{-1}{2}$
$\therefore x \in \mathbb{R}, x \neq -2$

3. We want $x^2e^{-x} < |x-1|$. Using GC,



$x > 1.50$ or $-0.883 < x < 0.739$.

4.



$x > 0.443$.

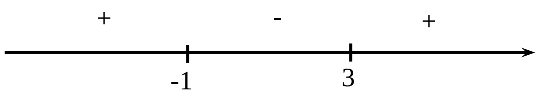
From $|x-2| < e^x$, replace x by $x+1$ to obtain:

$$|x+1-2| < e^{x+1}$$
$$\frac{|x-1|}{e} < e^x$$

$$\therefore x+1 > 0.443$$
$$x > -0.557$$

5.

$$x \neq 3$$
$$\frac{4x}{x-3} - 1 \geq 0$$
$$\frac{4x}{x-3} - \frac{x-3}{x-3} \geq 0$$
$$\frac{4x-x+3}{x-3} \geq 0$$
$$\frac{3x+3}{x-3} \geq 0$$
$$\frac{x+1}{x-3} \geq 0$$



$\therefore x \leq -1$ or $x \geq 3$
Since $x \neq 3$, $x \leq -1$ or $x > 3$

From $\frac{4x}{x-3} \geq 1$, replace x by $|x|$,

$$\frac{4|x|}{|x|-3} \geq 1$$

$|x| \leq -1$ (N.A.) or $|x| > 3$
$\implies x < -3$ or $x > 3$