1. (a)

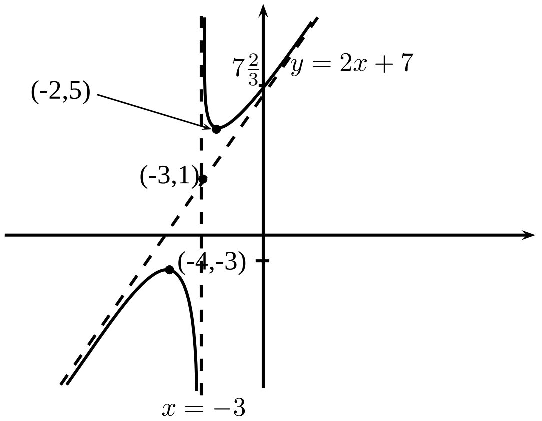
$$y = \frac{2x^2+13x+23}{x+3}$$
$$y(x+3) = 2x^2+13x+23$$
$$yx+3y-2x^2-13x-23=0$$
$$-2x^2+x(-13+y)+3y-23=0$$

We want to obtain values of y for which the curve has no solutions. Therefore,

$$b^2-4ac<0$$
$$(-13+y)^2-4(-2)(3y-23)<0$$
$$169-26y+y^2+24y-184<0$$
$$y^2-2y-15<0$$
$$(y-5)(y+3)<0$$
$$-3<y<5$$

(b)
By long division, $y = \frac{(x+3)(2x+7)+2}{x+3} = 2x+7+\frac{2}{x+3}$
Asymptotes:
$y = 2x+7$,
$x=-3$.

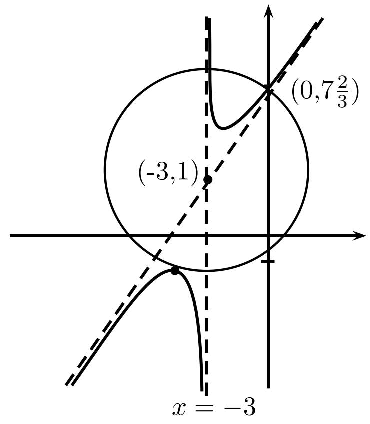
(c)



(d)
Equation of circle with radius k, centre $(-3,1)$:

$$(x+3)^2+(y-1)^2=k^2.$$

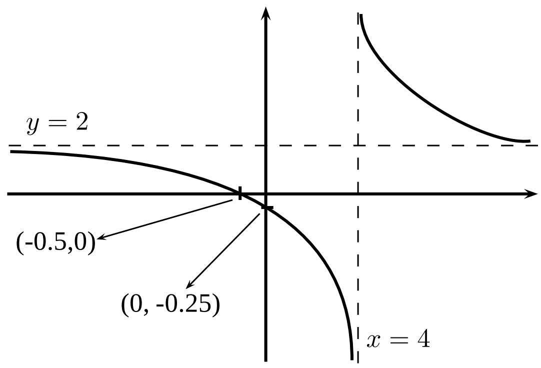
Intersecting with $y=\frac{2x^2+13x+23}{x+3}$, we get
$(x+3)^2+(\frac{2x^2+13x+23}{x+3}-1)^2=k^2$
$(x+3)^2+(\frac{2x^2+12x+20}{x+3})^2=k^2$



For the intersection to have positive root, k must be bigger than "distance from $(-3,1)$ to $(0,7\frac{2}{3})$"

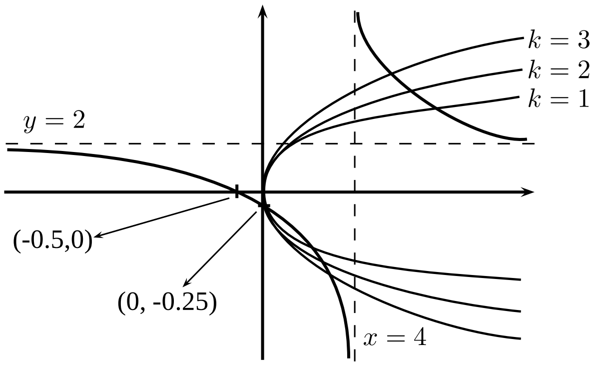
$$k^2 > (-3-0)^2+\left(1-7\frac{2}{3}\right)^2$$
$$k^2 > 9+\frac{400}{9}$$
$$k^2 > \frac{481}{9}$$
$$k^2-\frac{481}{9}>0$$
$$\left(k-\frac{\sqrt{481}}{3}\right)\left(k+\frac{\sqrt{481}}{3}\right)>0$$
$$k<-\frac{\sqrt{481}}{3} \text{ or } k>\frac{\sqrt{481}}{3}$$

2. (a)
Asymptotes:
$y=2$
$x=4$



(b) Intersecting $y^2=kx$ with $y=\frac{2x+1}{x-4}$ will give us

$$\left(\frac{2x+1}{x-4}\right)^2=kx.$$

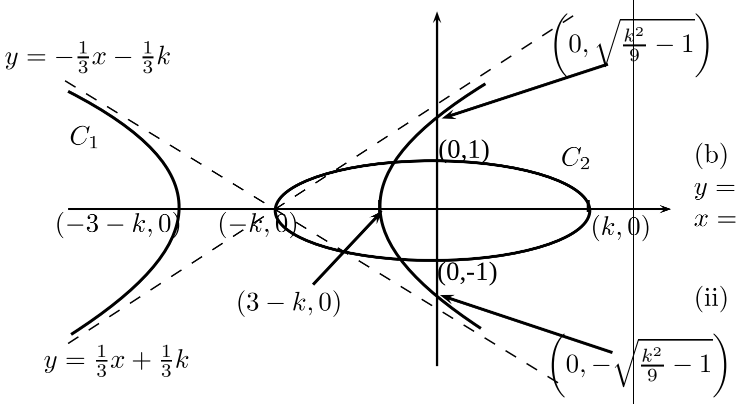


Since k is a positive integer, the minimum value of k is 1.
From the sketches above, we see that $y^2 = kx$ will cut the other graph at 3 points as long as $k \geq 1$.

3. (i)
$C_1 : \frac{(x+k)^2}{9} - y^2 = 1$

Asymptotes:
$\frac{y}{x+k} = \frac{\pm 1}{3}$
$\implies y = \frac{1}{3}x + \frac{1}{3}k$ and $y = \frac{-1}{3}x - \frac{1}{3}k$

x-intercepts: $(3-k, 0), (-3-k, 0)$
y-intercepts: $\left(0, -\sqrt{\frac{k^2}{9} - 1}\right), \left(0, \sqrt{\frac{k^2}{9} - 1}\right)$



(ii)
$\frac{x^2}{a^2} + \frac{(x+k)^2-9}{9} = 1$
$\frac{x^2}{a^2} + \frac{(x+k)^2}{9} - 1 = 1$
$\frac{(x+k)^2}{9} - 1 = 1 - \frac{x^2}{a^2}$

Solution to this equation is the intersection of:
Hyperbola C_1: $y^2 = \frac{(x+k)^2}{9} - 1$
Ellipse: $y^2 = 1 - \frac{x^2}{a^2}$

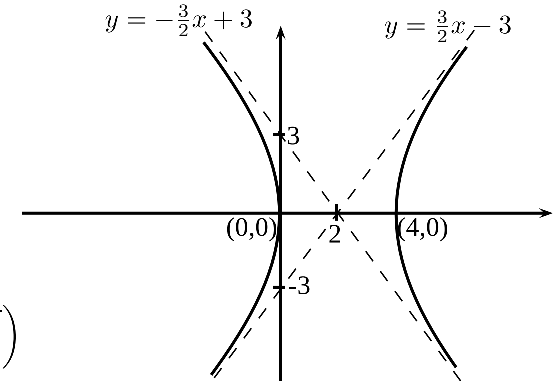
From the graph, to have 4 real solutions, the 'breadth' of the ellipse must have length at least $k+3$ in order to cut the 'left' side of the hyperbola.
$\therefore a > k+3$

4. (a)

$$9x^2 - 4y^2 - 36x = 0$$
$$9(x^2 - 4x) - 4y^2 = 0$$
$$9((x-2)^2 - 2^2) - 4y^2 = 0$$
$$9(x-2)^2 - 4y^2 = 36$$
$$\frac{(x-2)^2}{4} - \frac{y^2}{9} = 1$$
$$\frac{(x-2)^2}{2^2} - \frac{y^2}{3^2} = 1$$

Asymptotes:
$\frac{y}{x-2} = \pm\frac{3}{2}$
$\implies y = \frac{3}{2}x - 3$ or $y = \frac{-3}{2}x + 3$

Intercepts: $(4,0), (0,0)$



(b) (i)
$y = \frac{x}{2}$
$x = 1$

(ii)

$$\frac{dy}{dx} = \frac{1}{2} - A(x-1)^{-2}$$

$$\frac{dy}{dx} = 0$$
$$\frac{1}{2} - \frac{A}{(x-1)^2} = 0$$
$$(x-1)^2 - 2A = 0$$
$$(x-1)^2 = 2A$$
$$x^2 - 2x + 1 - 2A = 0$$

Since the curve has no stationary points,

$$b^2 - 4ac < 0$$
$$(-2)^2 - 4(1)(1-2A) < 0$$
$$4 - 4 + 8A < 0$$
$$8A < 0$$
$$A < 0$$

5. (a)
$\frac{dy}{dx} = \frac{(2x-a)(x-2)-(x^2-ax+b)}{(x-2)^2}$

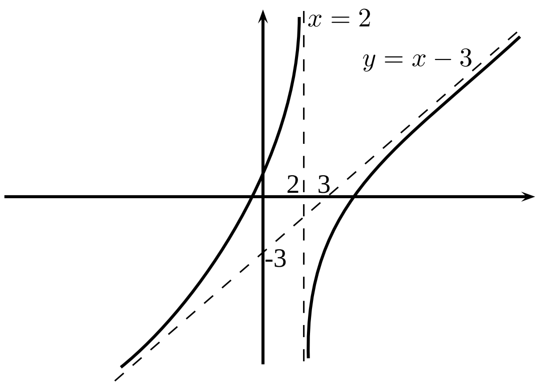
$$\begin{aligned}\frac{dy}{dx} &= 0\\(2x-a)(x-2)-(x^2-ax+b) &= 0\\2x^2-4x-ax+2a-x^2+ax-b &= 0\\x^2+x(-a+a-4)+2a-b &= 0\end{aligned}$$

Since the curve has stationary points,

$$\begin{aligned}b^2-4ac &> 0\\(-4)^2-4(1)(2a-b) &> 0\\16-8a+4b &> 0\\4-2a+b &> 0\\b &> 2a-4\end{aligned}$$

(b)
When $a=5$, $y=\frac{x^2-5x+b}{x-2}$.
By long division, $y = x-3+\frac{b-6}{x-2}$.

Asymptotes:
$y = x-3$,
$x = 2$.



$$\begin{aligned}2x^2-10x-1 &= (mx-3)(2x-4)\\x^2-5x-0.5 &= \frac{(mx-3)(2x-4)}{2}\\x^2-5x-0.5 &= (mx-3)(x-2)\\\frac{x^2-5x-0.5}{x-2} &= mx-3\end{aligned}$$

Therefore, we want the intersection of $y=\frac{x^2-5x-0.5}{x-2}$ with $y=mx-3$.

For the graph to have 2 real roots, $m<1$. (We compare the gradient of the asymptote $y=x-3$ with our graph $y=mx-3$)

6. a(i)

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x}{1+\sin x}\\(1+\sin x)\frac{dy}{dx} &= \cos x\\e^y\frac{dy}{dx} &= \cos x\end{aligned}$$

a(ii)

Differentiate w.r.t x,

$$\begin{aligned}e^y\left(\frac{dy}{dx}\right)^2+e^y\frac{d^2y}{dx^2} &= -\sin x\\e^y\left[\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}\right] &= -\sin x\end{aligned}$$

Differentiate w.r.t x,

$$e^y\frac{dy}{dx}\left[\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}\right]+e^y\left[2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{d^3y}{dx^3}\right] = -\cos x$$

$$e^y\left(\frac{dy}{dx}\right)^3+3e^y\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+e^y\frac{d^3y}{dx^2} = -e^y\frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)^3+3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{d^3y}{dx^3}+\frac{dy}{dx} = 0 \text{ (shown)}$$

When $x=0$,

$y=0$

$\frac{dy}{dx}=1$

$\frac{d^2y}{dx^2}=-1$

$\frac{d^3y}{dx^3}=1$

$$\begin{aligned}\therefore y &= x+\frac{-1}{2}x^2+\frac{1}{3!}x^3+\cdots\\&= x-\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots\end{aligned}$$

(c)

$\frac{\cos x}{1+\sin x} = \frac{dy}{dx} \approx 1-x+\frac{1}{2}x^2+\cdots$

7. (a)

$\frac{dy}{dx} = \frac{2}{1+2x}$

$(1+2x)\frac{dy}{dx} = 2$

(b)

Differentiate w.r.t x,

$2\frac{dy}{dx}+(1+2x)\frac{d^2y}{dx^2} = 0$

Differentiate w.r.t x,

$2\frac{d^2y}{dx^2}+2\frac{d^2y}{dx^2}+(1+2x)\frac{d^3y}{dx^3} = 0$

When $x=0$,

$y = 0$

$\frac{dy}{dx} = 2$

$\frac{d^2y}{dx^2} = -4$

$\frac{d^3y}{dx^3} = 16$

$$\therefore y = 2x + \frac{-4}{2!}x^2 + \frac{16}{3!}x^3 + \cdots$$
$$= 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

(c)

$$\ln\sqrt{\frac{1+2x}{1-2x}} = \frac{1}{2}[\ln(1+2x) - \ln(1-2x)]$$
$$= \frac{1}{2}\left[2x - 2x^2 + \frac{8}{3}x^3 - (-2x - 2(-x)^2 + \frac{8}{3}(-x)^3) + \cdots\right]$$
$$= \frac{1}{2}\left(4x + \frac{16}{3}x^3 + \cdots\right)$$
$$= 2x + \frac{8}{3}x^3 + \cdots$$
$$\therefore p = 2, \quad q = \frac{8}{3}$$