Complex Numbers Revision 2

1. i. The complex numbers z_1 and z_2 are given by $z_1 = -1 + ia$ and $z_2 = \sqrt{2} - \sqrt{2}i$ where a is a real number. Given that $\text{Im}\left(\frac{z_1}{z_2}\right) = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$, show that $a = \sqrt{3}$. [2]

   ii. State the modulus and argument of each of z_1 and z_2. [2]

   iii. Sketch an Argand diagram with origin O, showing the points P, Q and R representing the complex numbers z_1, z_2 and $(z_1 + z_2)$ respectively. Give a geometrical description of $OPRQ$. [2]

   iv. Deduce from your diagram that $\tan \frac{5\pi}{24} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$. [3]

   [(ii) $|z_1| = 2, |z_2| = 2$; $\arg(z_1) = \frac{2\pi}{3}, \arg(z_2) = \frac{-\pi}{4}$ (iii) parallelogram]

2. [2011/AJC/Prelim/13]

   i. The complex numbers p and q satisfy the simultaneous equations

   $p^* + 10i = qi + 5$

   $|p|^2 - q - 5 + 2i = 0$

   Given that $\text{Im}(p) < 0$, find p in the Cartesian form. [3]

   ii. Hence find the values of n for which p^{2n} is purely imaginary. [2]

   iii. The complex number w is such that $\arg(\frac{w}{p} - p^*) = -\frac{\pi}{2}$ and $w + w^* = -2$. Find w. [4]

   [(i) $p = 3 - 3i, q = 13 + 2i$ (ii) $n = 2k + 1$, where $k \in \mathbb{Z}$ (iii) $w = -1 - 19i$]

3. [2013/SAJC/I/11b]

   (a) Given that $z = p$ is a solution of the equation

   $$az^3 + bz^2 + cz + d = 0,$$

   where a and c are real constants while b and d are purely imaginary constants, show algebraically that $z = -p^*$ is another solution.

   [You may use the result $(z + w)* = z * +w^*$.] [3]

   (b) Hence, without using a calculator, solve the equation

   $$z^3 - 11iz^2 - 64z + 170i = 0,$$

   given that $z = 5 + 3i$ is one of the solutions. [3]

   [(b) $z = 5 + 3i, -5 + 3i, 5i$]

4. [2013/RVHS/I/7]

   The complex numbers w and z satisfy the equations

   $$\frac{w}{2} = \frac{3}{2}i$$

   $$z^* - 2w = 4 + 4i$$

   Find w and z. [4]

   [$z = -2 + 2i, w = -3 - 3i$]

5. [2015/ACJC/Prelim/I/11]

   The complex numbers p and q are given by $k + 2\mathbf{i}$ and $3 - 3\mathbf{i}$ respectively, where $k \in \mathbb{R}, k > 0$.

   (a) $P(x)$ is a polynomial of degree n with real coefficients where the coefficient of x^n is 1. Given that p and q are roots of $P(x) = 0$, state the least possible value of n. For this value of n, express $P(x)$ as a product of quadratic factors with real coefficients. [3]

(b) The complex number $\frac{iq^2}{2p}$ has modulus $\frac{9}{4}$ and argument α, where $-\pi < \alpha \leq \pi$. Without using a calculator, find the exact values of k and α. [4]

[(a) $P(x) = (x^2 - 2kx + k^2 + 4)(x^2 - 6x + 18)$ (b) $k = 2\sqrt{3}$, $\alpha = -\frac{\pi}{6}$]