1. (i)

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{-1+ia}{\sqrt{2}-\sqrt{2}i}\\ &= \frac{(-1+ia)(\sqrt{2}+\sqrt{2}i)}{2+2}\\ &= \frac{1}{4}\left(-\sqrt{2}-\sqrt{2}i+\sqrt{2}ai-\sqrt{2}a\right)\\ &= \frac{1}{4}\left(-\sqrt{2}-\sqrt{2}a+i\left(-\sqrt{2}+\sqrt{2}a\right)\right)\end{aligned}$$

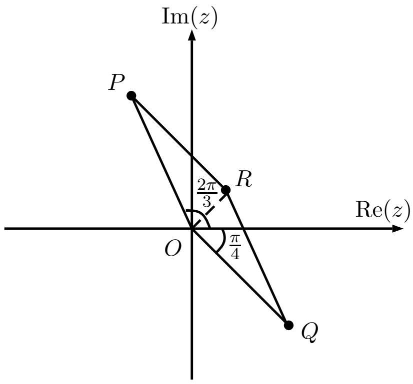
$$\begin{aligned}\therefore \frac{1}{4}\left(-\sqrt{2}+\sqrt{2}a\right) &= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)\\ \Rightarrow \sqrt{2}a &= \sqrt{6}\\ a &= \sqrt{3}\end{aligned}$$

(ii)

$|z_1| = \sqrt{1^2+3} = 2, \quad \arg(z_1) = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \frac{2\pi}{3}$

$|z_2| = \sqrt{2+2} = 2, \quad \arg(z_2) = -\tan^{-1}\left|\frac{-\sqrt{2}}{\sqrt{2}}\right| = -\frac{\pi}{4}$

(iii)



$OPRQ$ is a parallelogram.

(iv)

We now find the argument of $z_1 + z_2$.

Note that OR bisects $\angle POQ$.

$$\begin{aligned}\therefore \angle POR &= \frac{1}{2}\angle POQ\\ &= \frac{1}{2}\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)\\ &= \frac{11\pi}{24}\end{aligned}$$

$$\begin{aligned}\Rightarrow \ \arg\,(z_1+z_2) &= \frac{2\pi}{3} - \frac{11\pi}{24}\\ &= \frac{5\pi}{24}\end{aligned}$$

We first obtain $z_1 + z_2$ in cartesian form.

$$z_1 + z_2 = -1 + \sqrt{2} + i\left(\sqrt{3} - \sqrt{2}\right)$$

$$\text{Hence, } \tan\left(\frac{5\pi}{24}\right) = \frac{\text{Im }(z_1 + z_2)}{\text{Re }(z_1 + z_2)} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1}$$

2. a(i)

$p^* + 10i = qi + 5$ —— (1)

$|p|^2 - q - 5 + 2i = 0$

$q = |p|^2 - 5 + 2i$ —— (2)

Sub (2) into (1),

$p^* + 10i = (|p|^2 - 5 + 2i)i + 5$

Let $p = x + iy$,

$x - iy + 10i = (x^2 + y^2 - 5 + 2i)i + 5$

$x + i(10 - y) = (x^2 + y^2 - 5)i + 3$

Comparing real parts.

$x = 3$

Comparing imaginary parts,

$$\begin{aligned} 10 - y &= 3^2 + y^2 - 5 \\ 0 &= y^2 + y - 6 \\ y &= -3 \text{ or } 2 \text{ (rejected)} \end{aligned}$$

$\therefore p = 3 - 3i$

Sub p into (2),

$q = 3^2 + 3^2 - 5 + 2i = 13 + 2i$

(ii)

$$\begin{aligned} p^{2n} &= (3 - 3i)^{2n} \\ &= \left[\sqrt{3^2 + 3^2} e^{i\left(-\frac{\pi}{4}\right)}\right]^{2n} \\ &= 18^n e^{i\left(-\frac{n\pi}{2}\right)} \\ &= 18^n \left[\cos\left(-\frac{n\pi}{2}\right) + i\sin\left(-\frac{n\pi}{2}\right)\right] \end{aligned}$$

For the above to be purely imaginary,

$$\cos\left(-\frac{n\pi}{2}\right) = 0$$
$$\cos\left(\frac{n\pi}{2}\right) = 0$$
$$\frac{n\pi}{2} = \frac{\pi}{2} + k\pi$$
$$n = 1 + 2k, k \in \mathbb{Z}.$$

(iii)

$$w + w^* = -2$$
$$2\text{Re}(w) = -2$$
$$\text{Re}(w) = -1$$

Let $w = -1 + iy$

$$\begin{aligned}\frac{w}{p} - p^* &= \frac{-1+iy}{3-3i} - (3+3i)\\ &= \frac{(-1+iy)(3+3i)}{3^2+3^2} - (3+3i)\\ &= \frac{-3+3yi-3i-3y}{18} - 3 - 3i\\ &= \frac{-3-3y+i(3y-3)}{18} - 3 - 3i\\ &= \frac{-1-y+i(y-1)}{6} - 3 - 3i\\ &= \frac{-19-y+i(y-19)}{6}\end{aligned}$$

Since $\arg\left(\frac{w}{p} - p^*\right) = -\frac{\pi}{2}$, we deduce that $\text{Re}(\frac{w}{p} - p^*) = 0$

$$\therefore \frac{-19-y}{6} = 0$$
$$-19 - y = 0$$
$$y = -19$$

$\therefore w = -1 - 19i$

3. (a)

Since $z = p$ is a solution of the equation $az^3 + bz^2 + cz + d = 0$,

$$ap^3 + bp^2 + cp + d = 0$$

Taking complex conjugates on both sides and using the properties of complex conjugates,

$$(ap^3 + bp^2 + cp + d)^* = 0^*$$
$$(ap^3) + (bp^2)^* + (cp)^* + d^* = 0^*$$
$$a^*(p^*)^3 + b^*(p*)^2 + c^*p^* + d^* = 0^*$$

Since $0, a$ and c are real, $0^* = 0, a^* = a, c^* = c$

Since b and d are purely imaginary, $b^* = -b, d^* = -d$.

Hence,

$$a(p^*)^3 - b(p^*)^2 + cp^* - d = 0$$
$$-(a(p^*)^3 - b(p^*)^2 + cp^* - d) = 0$$
$$a(-p^*)^3 + b(-p^*)^2 + c(-p^*) + d = 0$$

Therefore, $z = -p^*$ is another solution of the equation.

b(ii)

For $z^3 - 11iz^2 - 64z + 170i = 0$

Since $z = 5 + 3i$ is one of the solutions, by using (i), we conclude that $z = -(5 - 3i) = -5 + 3i$ is another solution.

Let q be the final solution of the equation, We then have

$$\begin{aligned}
&z^3 - 11iz^2 - 64z + 170i \\
&= (z - (5 + 3i))(z - (-5 + 3i))(z - q) \\
&= ((z - 3i) - 5)((z - 3i) + 5))(z - q) \\
&= ((z - 3i)^2 - 5^2)(z - q) \\
&= (z^2 + 6iz - 9 - 25)(z - q) \\
&= (z^2 - 6iz - 34)(z - q)
\end{aligned}$$

By comparing the constant term, $34q = 170i$, and thus $q = 5i$.

Therefore, the roots are $z = 5 + 3i, -5 + 3i, 5i$.

4. $\frac{w}{z} = \frac{3}{2}i - - - -(1)$

$z^* - 2w = 4 + 4i - - - -(2)$

$(1) \Rightarrow w = \frac{3}{2}iz$

Substitute into (2):

$z^* - 2\left(\frac{3}{2}iz\right) = 4 + 4i$

Let $z = x + iy$

$(x + iy)^* - 3i(x + iy) = 4 + 4i$

Comparing real and imaginary parts,

$x + 3y = 4$ and $-y - 3x = 4$

Solving,

$x = -2$ and $y = 2$.

$\therefore z = -2 + 2i$

$w = -3 - 3i$

5. (a)

Since $P(x)$ has only real coefficients and p and q are complex roots of $P(x) = 0$, then p^* and q^* are also complex roots of $P(x) = 0$.

Hence least $n = 4$.

$$\begin{aligned}
P(x) &= (x - p)(x - p^*)(x - q)(x - q^*) \\
&= [(x - k) - 2i][(x - k) + 2i][(x - 3) + 3i][(x - 3) - 3i] \\
&= [(x - k)^2 - (2i)^2][(x - 3)^2 - (3i)^2] \\
&= (x^2 - 2kx + k^2 + 4)(x^2 - 6x + 18)
\end{aligned}$$

b(i)

$p = k + 2i \Rightarrow |p| = \sqrt{k^2 + 4}$

$q = 3 - 3i \Rightarrow |q| = \sqrt{3^2 + 3^2} = 3\sqrt{2}; \quad \arg(q) = -\tan^{-3}\left(\frac{1}{3}\right) = -\frac{\pi}{4}$

$\left|\frac{iq^2}{2p}\right| = \frac{|q|^2}{2|p|} = \frac{(3\sqrt{2})^2}{2\sqrt{k^2+4}}$

$$\frac{9}{4} = \frac{9}{\sqrt{k^2+4}}$$

$$16 = k^2 + 4$$

$$k = \pm\sqrt{16-4}$$

$k = 2\sqrt{3}$ or $k = -2\sqrt{3}$ (NA since $k > 0$)

When $k = 2\sqrt{3}$,

$$\alpha = \arg\left(\frac{iq^2}{2p}\right)$$

$$= \arg(i) + \arg(q^2) - \arg(2p)$$

$$= \frac{\pi}{2} + 2\left(-\frac{\pi}{4}\right) - \frac{\pi}{6}$$

$$= -\frac{\pi}{6}$$

b(ii)

$$iz^4 = -\frac{q^2}{2p}$$

$$z^4 = -\frac{q^2}{2p}\left(\frac{1}{i}\right)\left(\frac{i}{i}\right) = -\frac{iq^2}{2p} = \frac{9}{4}c^{\left(\frac{\pi}{6}+2k\pi\right)}, k = 0, \pm 1, 2$$

$$z = \left(\frac{9}{4}\right)e^{\left(\frac{\pi}{24}+\frac{k\pi}{2}\right)i}, k = 0, \pm 1, 2$$

$$z = \sqrt{\frac{3}{2}}e^{\left(\frac{13\pi}{2}\right)i}, \sqrt{\frac{3}{2}}e^{\left(\frac{pi}{24}\right)i}, \sqrt{\frac{3}{2}}e^{\left(\frac{11\pi}{24}\right)i}, \sqrt{\frac{3}{2}}e^{\left(\frac{23\pi}{24}\right)i}$$