Differentiation J1 Revision

1. [2011/YJC/Prelim/I/8]

   (a) A wire of length 40 cm is bent to form an isoceles triangle ABC with $AB = AC$ and $BC = x$ cm.

      i. Show that the are z cm^2 of the triangle ABC can be expressed as

      $$z = x\sqrt{100 - 5x}.$$

      [2]

      ii. Hence prove algebraically that the area of the triangle is maximised when the triangle is equilateral. [5]

   (b) The equation of a curve C is $x^3 + 2y^3 + 3xy = k$, where k is a constant.

      i. Find $\frac{dy}{dx}$ in terms of x and y. [3]

      ii. It is given that C has a tangent which is parallel to the x-axis. Show that the x-coordinate of the point of contact of the tangent with C must satisfy

      $$2x^6 + 2x^3 + k = 0.$$

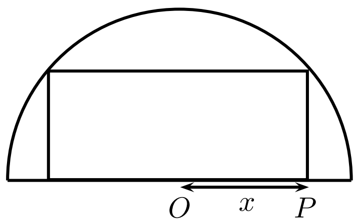
      [3]

      iii. Hence, find the values of k when the line $y = -1$ is a tangent to the curve C. [2]

   [b(i) $\frac{dy}{dx} = \frac{-y-x^2}{2y^2+x}$ b(iii) $k = 0, k = -4$]

2. [2010/JJC/Prelim/II/1]

   (a)

   The diagram shows a rectangle inscribed in a semicircle of centre, O, and fixed radius a. The length OP is denoted by x. Show that, as x varies, the perimeter of the rectangle is a maximum when its sides are in the ratio 4 : 1. [6]

   (b) Variables x and y are related by the equation

   $$y^2 + xy = x^2 - \frac{2}{x} + 3, \text{ where } y > 0$$

   Given that x is increasing at the rate of $\frac{1}{5}$ units s^{-1}, find the rate of increase of y when x is 1. [4]

   [(b) $\frac{1}{5}$ units s^{-1}]

3. [2011/SRJC/Prelim/II/3]

   A curve C is defined by the parametric equations

   $$x = \theta - \cos\theta,\ y = \theta + \cos\theta,\ 0 \le \theta \le 2\pi$$

   (a) Find the equation of the tangent to the curve C that is parallel to the y-axis. [3]

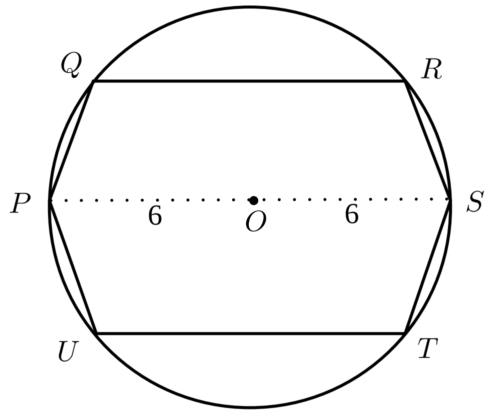
   (b) Find the exact equation of the tangent to the curve at the point P where $\theta = \frac{\pi}{6}$. [3]

   (c) Hence, determine whether the tangent at point P will meet the curve C again. [3]

   [(a) $x = \frac{3\pi}{2}$ (b) $y = \frac{1}{3}x + \frac{\pi}{9} + \frac{2}{\sqrt{3}}$ (c) yes]

4. [2010/DHS/Prelim/I/8]

The diagram shows a hexagon $PQRSTU$ inscribed in a circle wih radius 6 cm. The sides QR and UT are parallel, and $QR = UT = 2x$ cm.



(a) Show that A, the area of the hexagon $PQRSTU$, is $2(6+x)\sqrt{36-x^2}$ cm^2. [3]

(b) Using differentiation, find the value of x when A is maximum.

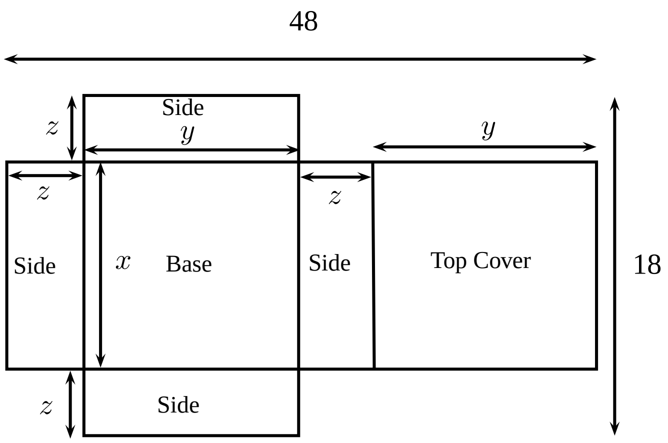
(You need not verify that it gives a maximum value.) [4]

(c) Initially $x = 6$ cm, and the lengths of the parallel sides QR and UT are each decreasing at a constant rate of $\frac{1}{10}$ cm s^{-1}. Find the rate of change of A at the instant when $x = 2$ cm. [3]

[Area of a trapezium= $\frac{1}{2}\times$ sum of the lengths of the parallel sides $\times$ height]

[(b) $x = 3$ cm (c) $\frac{-\sqrt{2}}{5}$ cm s^{-1}]

5. [2011/AJC/Prelim/I/7]



The above 48 cm by 18 cm plastic sheet is cut out to form a shape represented by the shaded region. The shape is folded into a box with width x, length y and height z as shown in the diagram.

i. Express the volume of the box, V, in terms of y. [2]

ii. Using differentiation, find the maximum possible volume of the box. [3]

[(i) $V = -2y^3 + 78y^2 - 720y$ a(ii) 800 cm^3]

6. [2011/SAJC/Prelim/I/12]

   (a) A curve has parametric equations

   $$x = \theta - \sin\theta,\ \ y = 1 - \cos\theta,\ \text{ for } 0 < \theta < 2\pi.$$

   Find the equation of the tangent parallel to the x-axis. [3]

   (b) The normal to the curve at the point with parameter $\frac{2\pi}{3}$, meets the x- and y-axes at P and Q respectively.

   Show that the equation of the normal is $y = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$.

   Hence find the area of the triangle OPQ. [5]

   (c) Given that θ is increasing at a rate of 2 radians per second, find the rate of change for $\frac{dy}{dx}$ of the curve at $\theta = \frac{\pi}{3}$. [3]

   [Hint: rate of change of $\frac{dy}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dt}$]

   [(a) $y = 2$ (b) $\frac{2\sqrt{3}\pi^2}{9}$ units^2 (c) -4 units/s]

7. [2011/RI/Prelim/II/3]

   A curve C has parametric equations

   $$x = a\sin^4\theta,\ \ y = b\cos^4\theta,$$

   where a and b are positive constants and $0 < \theta < \frac{\pi}{2}$.

   i. Sketch the curve C in the case where $a = 2$ and $b = 1$. [1]

   The tangent to C at the point $(a\sin^4\theta, b\cos^4\theta)$ cuts the x-axis at the point P and the y-axis at the point Q.

   ii. Show that, at the point $(a\sin^4\theta, b\cos^4\theta)$ on C, $\frac{dy}{dx} = -\frac{b}{a}\cot^2\theta$. [3]

   iii. Show that the coordinates of P are $(a\sin^2\theta,\ 0)$ and find the coordinates of Q, simplifying your answer. [3]

   iv. The area of triangle OPQ is denoted by A, where O refers to the origin. The value of A varies as θ varies between 0 and $\frac{\pi}{2}$. Determine the exact value of θ at which A attains a maximum, and write down the corresponding value of A. [3]

   (It is not necessary to verify the nature of the stationary value of A.)

   [(iii) $(0, b\cos^2\theta)$ (iv) A is maximum when $\theta = \frac{\pi}{4}$, $A = \frac{ab}{8}$]

8. [2009/CJC/Prelim/I/6]

   Given that $y = e^{1+x^2}$, prove that

   $$\frac{d^2y}{dx^2} = 2\left(y + x\frac{dy}{dx}\right).$$

9. [2010/SAJC/Prelim/I/9]

   It is given $\frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2}$, where $\tan^{-1}x$ denotes the principal value.

   Show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$.