1. (i)

$$3x^2+6y^2\frac{dy}{dx}+3\left(y+x\frac{dy}{dx}\right)=0$$
$$3x^2+6y^2\frac{dy}{dx}+3y+3x\frac{dy}{dx}=0$$
$$\frac{dy}{dx}(6y^2+3x)=-3x^2-3y$$
$$\frac{dy}{dx}=\frac{-3x^2-3y}{6y^2+3x}$$
$$=\frac{-x^2-y}{2y^2+x}$$

(ii)

parallel to $x$-axis,

$$\frac{dy}{dx}=0$$
$$-x^2-y=0$$
$$y=-x^2$$

Substitute $y=-x^2$ into C,

$$x^3+2(-x^2)^3+3x(-x^2)=k$$
$$x^3-2x^6-3x^3-k=0$$
$$2x^6+2x^3+k=0$$

(iii)

when $y=-1$,

$$-1=-x^2$$
$$x^2=1$$
$$x=\pm1$$

when $x=1$

$$2(1)^6+2(1)^3+k=0$$
$$k=-4$$

when $x=-1$

$$2(-1)^6+2(-1)^3+k=0$$
$$k=0$$

2. (a)

$\frac{dx}{d\theta}=1+\sin\theta, \frac{dy}{d\theta}=1-\sin\theta$

$\frac{dy}{dx}=\frac{1-\sin\theta}{1+\sin\theta}$

Since the tangent is parallel to $y$-axis, its gradient is undefined, hence denominator $=0$.

$$1+\sin\theta=0$$
$$\sin\theta=-1$$
$$\theta=\frac{3\pi}{2}$$

Equation of tangent: $x=\frac{3\pi}{2}-\cos\frac{3\pi}{2}=\frac{3\pi}{2}$.

(b)

$\theta=\frac{\pi}{6}$,

$x=\frac{\pi}{6}-\cos\frac{\pi}{6}=\frac{\pi}{6}-\frac{\sqrt3}{2}$,

$y=\frac{\pi}{6}+\cos\frac{\pi}{6}=\frac{\pi}{6}+\frac{\sqrt3}{2}$

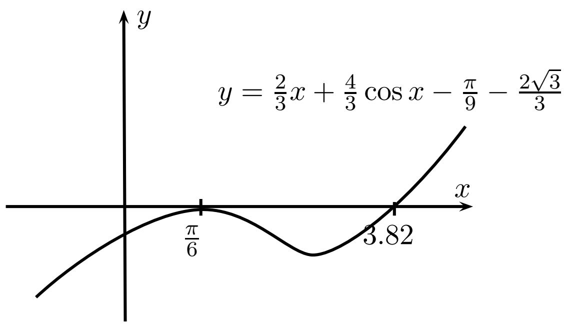
equation tangent:

$$\frac{y-(\frac{\pi}{6}+\frac{\sqrt3}{2})}{x-(\frac{\pi}{6}-\frac{\sqrt3}{2})}=\frac{1-\sin\frac{\pi}{6}}{1+\sin\frac{\pi}{6}}$$
$$y-\left(\frac{\pi}{6}+\frac{\sqrt3}{2}\right)=\frac{1}{3}\left(x-\left(\frac{\pi}{6}-\frac{\sqrt3}{2}\right)\right)$$
$$y=\frac{1}{3}x+\frac{\pi}{9}+\frac{2\sqrt3}{3}.$$

(c)

Substitute $C$ into the tangent,

$$\theta+\cos\theta=\frac{1}{3}\theta-\frac{1}{3}\cos\theta+\frac{\pi}{9}+\frac{2\sqrt3}{3}$$
$$\frac{2}{3}\theta+\frac{4}{3}\cos\theta=\frac{\pi}{9}+\frac{2\sqrt3}{3}$$




Since there are 2 solutions for $\theta$ in the range $0\le\theta\le2\pi$, tangent will meet the curve again.

3. (a) $\frac{dx}{d\theta}=1-\cos\theta$, $\frac{dy}{d\theta}=\sin\theta$

$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$

Equation of tangent parallel to x-axis, $\frac{dy}{dx}=0$,

$$\sin\theta=0$$
$$\theta=\pi$$

$$y=1-\cos\pi$$
$$=1-(-1)$$
$$=2$$

(b)

At $t=\frac{2\pi}{3}$,

$y=1-\cos\frac{2\pi}{3}=1\frac{1}{2}$

$x = \frac{2\pi}{3} - \sin\frac{2\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$
$\frac{dy}{dx} = \frac{\sin\frac{2\pi}{3}}{1-\cos\frac{2\pi}{3}} = \frac{\sqrt{3}}{3}$

Equation of normal:

$$\frac{y-\frac{3}{2}}{x-\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)} = -1 \div \frac{\sqrt{3}}{3}$$
$$= -\sqrt{3}$$
$$y - \frac{3}{2} = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3} - \frac{3}{2}$$
$$y = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$$

At P, $y = 0$,

$$0 = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$$
$$x = \frac{2\pi}{3}$$

At Q, $x = 0$

$$y = \frac{2\sqrt{3}\pi}{3}$$

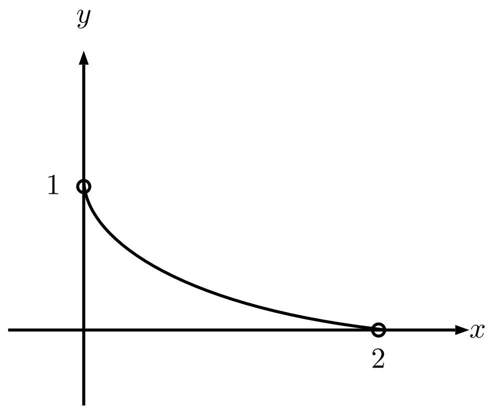
Area, $\frac{1}{2}(\frac{2\pi}{3})(\frac{2\sqrt{3}\pi}{3}) = \frac{2\sqrt{3}\pi^2}{9} units^2$

(c)

$\frac{d\theta}{dt} = 2$ , $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$
Rate of change of $\frac{dy}{dx}$ : $\frac{d(\frac{dy}{dx})}{dt}$
Note that $\frac{d(\frac{dy}{dx})}{dt} = \frac{d(\frac{dy}{dx})}{d\theta} \times \frac{d\theta}{dt}$
$\frac{d(\frac{dy}{dx})}{d\theta} = \frac{\cos\theta(1-\cos\theta)-\sin\theta(\sin\theta)}{(1-\cos\theta)^2}$

At $\theta = \frac{\pi}{3}$, $\frac{d(\frac{dy}{dx})}{d\theta} = \frac{\frac{1}{2}(\frac{1}{2})-\frac{3}{4}}{\frac{1}{4}}$=-2
$\therefore \frac{d(\frac{dy}{dx})}{dt} = -2 \times 2 = -4$units/s

4. (i)




(ii)

$$\frac{dx}{d\theta} = 4a\sin^3\theta\cos\theta$$
$$\frac{dy}{d\theta} = 4b\cos^3\theta(-\sin\theta)$$
$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$
$$= \frac{-4b\cos^3\theta\sin\theta}{4a\sin^3\theta\cos\theta}$$
$$= -\frac{b}{a}\frac{\cos^2\theta}{\sin^2\theta}$$
$$= -\frac{b}{a}\cot^2\theta$$

(iii)

Equation of tangent:

$y - b\cos^4\theta = -\frac{b}{a}\cot^2\theta(x - a\sin^4\theta)$

At $P$, $y = 0$,

$$-b\cos^4\theta = -\frac{b}{a}\cot^2\theta x + b\cot^2\theta\sin^4\theta$$
$$-b\cos^4\theta = -\frac{b}{a}\cot^2\theta x + b\cos^2\theta\sin^2\theta$$
$$-b\cos^2\theta = -\frac{b}{a}\frac{1}{\sin^2\theta}x + b\sin^2\theta$$
$$-b\cos^2\theta - b\sin^2\theta = -\frac{b}{a}\frac{1}{\sin^2\theta}x$$
$$-b = -\frac{b}{a}\frac{1}{\sin^2\theta}x$$
$$x = a\sin^2\theta$$
$$\therefore P(a\sin^2\theta, 0)$$

At $Q$, $x=0$,

$$\begin{aligned} y - b\cos^4\theta &= -\frac{b}{a}\cot^2\theta(0 - a\sin^4\theta) \\ y &= b\cos^4\theta + b\cos^2\theta\sin^2\theta \\ &= b\cos^2\theta(\cos^2\theta+\sin^2\theta) \\ &= b\cos^2\theta \\ \therefore\ & Q(0, b\cos^2\theta) \end{aligned}$$

(iv)

$$\begin{aligned} \text{Area } \triangle OPQ &= \frac{1}{2}(9\sin^2\theta)(b\cos^2\theta) \\ &= \frac{ab}{2}\sin^2\theta\cos^2\theta \\ &= \frac{ab}{2}(\sin\theta\cos\theta)^2 \\ &= \frac{ab}{2}\left(\frac{\sin 2\theta}{2}\right)^2 \\ &= \frac{ab}{8}\sin^2 2\theta \end{aligned}$$

$$\begin{aligned} \frac{dA}{d\theta} &= \frac{ab}{8}(2\sin 2\theta\cos 2\theta\cdot 2) \\ &= \frac{ab}{2}(\sin 2\theta\cos 2\theta) \\ &= \frac{ab}{4}\sin 4\theta \end{aligned}$$

$$\begin{aligned} \frac{dA}{d\theta} &= 0 \\ \Rightarrow \sin 4\theta &= 0 \\ 4\theta &= 0, \pi, 2\pi \\ \theta &= 0, \frac{\pi}{4}, \frac{\pi}{2} \end{aligned}$$

For $0 < \theta < \dfrac{\pi}{2}$, $\quad \theta = \dfrac{\pi}{4}$.

$$\begin{aligned} \frac{d^2A}{d\theta^2} &= \frac{ab}{4}(4\cos 4\theta) \\ &= ab\cos 4\theta \end{aligned}$$

When $\theta = \frac{\pi}{4}$, $\frac{d^2A}{d\theta^2} = ab\cos\pi = -ab < 0$,

since $a, b < 0$.

Hence $A$ is max at $\theta = \frac{\pi}{4}$,

$$\begin{aligned} A &= \frac{ab}{8}\left(\sin\frac{\pi}{2}\right)^2 \\ &= \frac{ab}{8}(1)^2 \\ &= \frac{ab}{8} \end{aligned}$$

5. $y = e^{1+x^2}$

Differentiate w.r.t. $x$,

$\frac{dy}{dx} = 2xe^{1+x^2}$

Differentiate w.r.t. $x$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2(e^{1+x^2} + x(2x)e^{1+x^2}) \\ &= 2e^{1+x^2} + 4x^2e^{1+x^2} \\ &= 2y + 2x\left(2xe^{1+x^2}\right) \\ &= 2y + 2x\left(\frac{dy}{dx}\right) \\ &= 2\left(y + x\frac{dy}{dx}\right) \text{ (shown)} \end{aligned}$$

6.

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{\tan^{-1}x}}{1+x^2} \\ (1+x^2)\frac{dy}{dx} &= e^{\tan^{-1}x} \end{aligned}$$

Differentiate w.r.t. $x$,

$$\begin{aligned} 2x\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2} &= \frac{1}{1+x^2}e^{\tan^{-1}x} \\ &= \frac{dy}{dx} \end{aligned}$$

$$(2x-1)\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2} = 0 \text{ (shown)}$$

7. (i)

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x}{1+\sin x} \\ (1+\sin x)\frac{dy}{dx} &= \cos x \\ e^y\frac{dy}{dx} &= \cos x \end{aligned}$$

(ii)

Differentiate w.r.t $x$,

$$\begin{aligned} e^y\left(\frac{dy}{dx}\right)^2 + e^y\frac{d^2y}{dx^2} &= -\sin x \\ e^y\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\right] &= -\sin x \end{aligned}$$

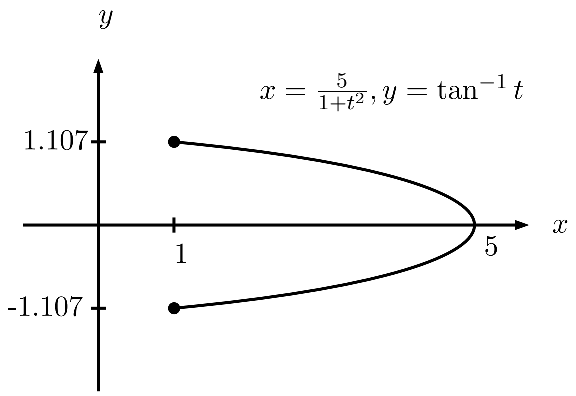
Differentiate w.r.t $x$,

$$e^y\frac{dy}{dx}\left[\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}\right]+e^y\left[2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{d^3y}{dx^3}\right]=-\cos x$$

$$e^y\left(\frac{dy}{dx}\right)^3+3e^y\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+e^y\frac{d^3y}{dx^2}=-e^y\frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)^3+3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{d^3y}{dx^3}+\frac{dy}{dx}=0 \text{ (shown)}$$

8. (a)




(b)

$$\frac{dx}{dt}=5(1+t^2)^{-2}(-1)(2t)$$
$$=\frac{-10t}{(1+t^2)^2}$$
$$\frac{dy}{dt}=\frac{1}{1+t^2}$$
$$\frac{dy}{dx}=\frac{1}{1+t^2}\times\frac{(1+t^2)^2}{-10t}$$
$$=\frac{-(1+t^2)}{10t}$$

At $t=1$,
$x=\frac{5}{1+1}=\frac{5}{2}$
$y=\tan^{-1}1=\frac{\pi}{4}$

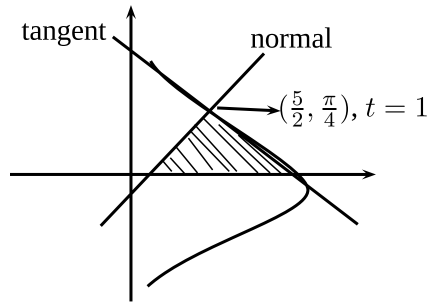
Equation of tangent:

$$\frac{y-\frac{\pi}{4}}{x-\frac{5}{2}}=\frac{-(1+1)}{10}$$
$$\frac{y-\frac{\pi}{4}}{x-\frac{5}{2}}=-\frac{1}{5}$$
$$5y-\frac{5\pi}{4}=\frac{5}{2}-x$$
$$5y+x=\frac{5}{2}+\frac{5\pi}{4}$$

Equation of normal:

$$\frac{y-\frac{\pi}{4}}{x-\frac{5}{2}}=\frac{10}{2}$$
$$\frac{y-\frac{\pi}{4}}{x-\frac{5}{2}}=5$$
$$y=5x-\frac{25}{2}+\frac{\pi}{4}$$

(c)




Intersection of tangent with x-axis:

$$x=\frac{5}{2}+\frac{5\pi}{4}$$

Intersection of normal with x-axis:

$$x=\frac{5}{2}-\frac{\pi}{20}$$

Area:

$$=\frac{1}{2}\left(\sqrt{\left(\frac{5}{2}-\frac{5}{2}-\frac{5\pi}{4}\right)}\right.$$
$$=\frac{1}{2}\left(\sqrt{\frac{25\pi^2}{16}+\frac{\pi^2}{16}}\right)\left(\right.$$
$$=\frac{1}{2}\left(\frac{\pi}{4}\sqrt{26}\right)\left(\frac{\pi}{20}\sqrt{26}\right)$$
$$=\frac{26\pi^2}{160}$$
$$=\frac{13\pi^2}{80}$$