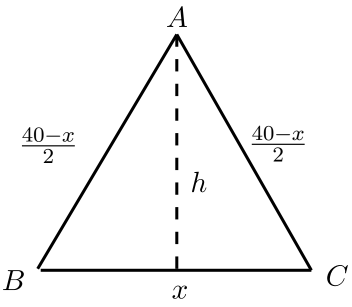
Differentiation Revision: Solution

1. a(i)



$$h^2 + \left(\frac{1}{2}x\right)^2 = \left(20 - \frac{x}{2}\right)^2$$
$$h^2 = \left(20 - \frac{x}{2}\right)^2 - \left(\frac{1}{2}x\right)^2$$
$$= 400 - 20x + \frac{x^2}{4} - \frac{x^2}{4}$$
$$= 400 - 20x$$
$$h = \sqrt{400 - 20x} = 2\sqrt{100 - 5x}$$

$$z = \tfrac{1}{2}x(2\sqrt{100 - 5x}) = x\sqrt{100 - 5x}$$

a(ii)

$$\frac{dz}{dx} = \sqrt{100 - 5x} + x\left(\frac{1}{2}\right)(100 - 5x)^{-\frac{1}{2}}(-5)$$
$$0 = \sqrt{100 - 5x} - \frac{5x}{2}(100 - 5x)^{-\frac{1}{2}}$$
$$0 = 100 - 5x - \frac{5x}{2}$$
$$75x = 100$$
$$x = \frac{40}{3}$$

x	$\left(\frac{40}{3}\right)^-$	$\frac{40}{3}$	$\left(\frac{40}{3}\right)^+$
$\frac{dz}{dx}$	+ve	0	-ve

$\therefore$ area of triangle is max when $x = \frac{40}{3}$, i.e., triangle is equilateral.

b(i)

$$3x^2 + 6y^2\frac{dy}{dx} + 3\left(y + x\frac{dy}{dx}\right) = 0$$
$$3x^2 + 6y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(6y^2 + 3x) = -3x^2 - 3y$$
$$\frac{dy}{dx} = \frac{-3x^2 - 3y}{6y^2 + 3x}$$
$$= \frac{-x^2 - y}{2y^2 + x}$$

b(ii)

parallel to x-axis,

$$\frac{dy}{dx} = 0$$
$$-x^2 - y = 0$$
$$y = -x^2$$

Substitute $y = -x^2$ into C,

$$x^3 + 2(-x^2)^3 + 3x(-x^2) = k$$
$$x^3 - 2x^6 - 3x^3 - k = 0$$
$$2x^6 + 2x^3 + k = 0$$

b(iii)

when $y = -1$,

$$-1 = -x^2$$
$$x^2 = 1$$
$$x = \pm 1$$

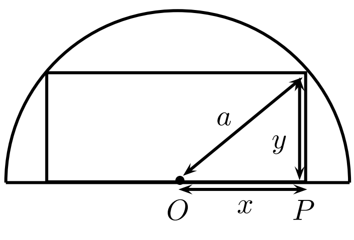
when $x = 1$

$$2(1)^6 + 2(1)^3 + k = 0$$
$$k = -4$$

when $x = -1$

$$2(-1)^6 + 2(-1)^3 + k = 0$$
$$k = 0$$

2.



$$y = \sqrt{a^2 - x^2}$$

perimeter, $P = 4x + 2y = 4x + 2\sqrt{a^2 - x^2}$

$$\frac{dP}{dx} = 4 + 2\left(\frac{1}{2}\right)(a^2 - x^2)^{-\frac{1}{2}}(-2x)$$
$$= 4 - 2x(a^2 - x^2)^{-\frac{1}{2}}$$

$$\frac{dP}{dx} = 0$$
$$0 = 4 - 2x(a^2 - x^2)^{-\frac{1}{2}}$$
$$0 = 4(a^2 - x^2)^{\frac{1}{2}} - 2x$$
$$2x = 4(a^2 - x^2)^{\frac{1}{2}}$$
$$4x^2 = 16a^2 - 16x^2$$
$$20x^2 = 16a^2$$
$$x^2 = \frac{4}{5}a^2$$
$$x = \frac{2}{\sqrt{5}}a$$

$$y = \sqrt{r^2 - \frac{4}{5}a^2}$$
$$= \sqrt{\frac{1}{5}a^2}$$
$$= \frac{1}{\sqrt{5}}a$$

$$2x : y$$
$$\implies \frac{4}{\sqrt{5}}a : \frac{1}{\sqrt{5}}a$$
$$\implies 4 : 1 \text{ (shown)}$$

(b)

Differentiate wrt t,

$$2y\frac{dy}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{-2(-1)}{x^2}\frac{dx}{dt} ----(1)$$

When $x = 1$,

$$y^2 + y = 1 - 2 + 3$$
$$y^2 + y - 2 = 0$$

$y = 1$ or $y = -2$ (rejected as $y > 0$)

Given $\frac{dx}{dt} = \frac{1}{5}$, from (1)

$$2\frac{dy}{dt} + \frac{1}{5} + \frac{dy}{dt} = 2\left(\frac{1}{5}\right) + 2\left(\frac{1}{5}\right)$$
$$3\frac{dy}{dt} = \frac{3}{5}$$
$$\frac{dy}{dt} = \frac{1}{5}$$

3. (a)

$\frac{dx}{d\theta} = 1 + \sin\theta$, $\frac{dy}{d\theta} = 1 - \sin\theta$

$\frac{dy}{dx} = \frac{1-\sin\theta}{1+\sin\theta}$

Since the tangent is parallel to y-axis, its gradient is undefined, hence denominator $= 0$.

$$1 + \sin\theta = 0$$
$$\sin\theta = -1$$
$$\theta = \frac{3\pi}{2}$$

Equation of tangent: $x = \frac{3\pi}{2} - \cos\frac{3\pi}{2} = \frac{3\pi}{2}$.

(b)

$\theta = \frac{\pi}{6}$,

$x = \frac{\pi}{6} - \cos\frac{\pi}{6} = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$,

$y = \frac{\pi}{6} + \cos\frac{\pi}{6} = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$

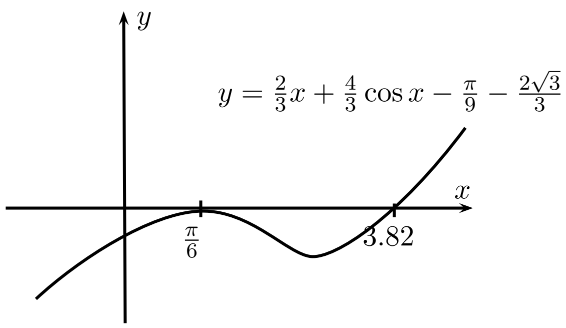
equation tangent:

$$\frac{y - (\frac{\pi}{6} + \frac{\sqrt{3}}{2})}{x - (\frac{\pi}{6} - \frac{\sqrt{3}}{2})} = \frac{1 - \sin\frac{\pi}{6}}{1 + \sin\frac{\pi}{6}}$$
$$y - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) = \frac{1}{3}\left(x - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)\right)$$
$$y = \frac{1}{3}x + \frac{\pi}{9} + \frac{2\sqrt{3}}{3}.$$

(c)

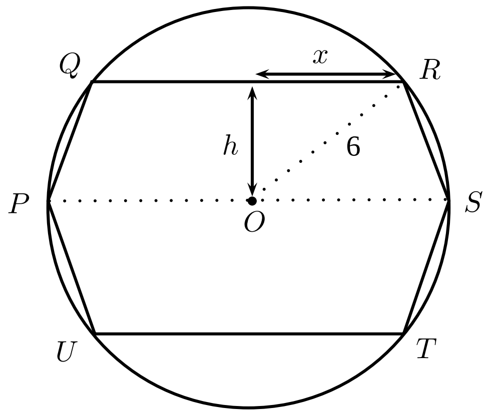
Substitute C into the tangent,

$$\theta + \cos\theta = \frac{1}{3}\theta - \frac{1}{3}\cos\theta + \frac{\pi}{9} + \frac{2\sqrt{3}}{3}$$
$$\frac{2}{3}\theta + \frac{4}{3}\cos\theta = \frac{\pi}{9} + \frac{2\sqrt{3}}{3}$$



Since there are 2 solutions for θ in the range $0 \le \theta \le 2\pi$, tangent will meet the curve again.

4. (a)



$$h = \sqrt{6^2 - x^2} = \sqrt{36 - x^2}$$

$$\text{Area} = 2\left[\frac{1}{2}(2x+12)\left(\sqrt{36-x^2}\right)\right]$$
$$= 2(x+6)\sqrt{36-x^2}$$

(b)

$$\frac{dA}{dx} = 0$$
$$2\left[(x+6)\left(\frac{1}{2}\left(36-x^2\right)^{-\frac{1}{2}}(-2x)\right) + \sqrt{36-x^2}\right] = 0$$
$$-x(x+6)(36-x^2)^{-\frac{1}{2}} + \sqrt{36-x^2} = 0$$
$$-x(x+6) + 36 - x^2 = 0$$
$$-2x^2 - 6x + 36 = 0$$
$$x^2 + 3x - 18 = 0$$
$$x = 3 \text{ or } -6 \text{ (rejected)}$$

(c)

Given that $\frac{d(QR)}{dt} = \frac{-1}{10} \Longrightarrow \frac{dx}{dt} = \frac{-1}{20}, x = 2.$

$$A = 2(x+6)\sqrt{36-x^2}$$
$$\frac{dA}{dt} = 2\left(\frac{dx}{dt}\sqrt{36-x^2} + (x+6)\left(\frac{1}{2}\right)(36-x^2)^{-\frac{1}{2}}(-2x)\frac{dx}{dt}\right)$$

Substitute $x = 2$, $\frac{dx}{dt} = \frac{-1}{20}$,

$$\frac{dA}{dt} = 2\left[\frac{-1}{20}\sqrt{32} + 4(32)^{-\frac{1}{2}}(-4)\left(\frac{-1}{20}\right)\right]$$
$$= 2\left[\frac{-4\sqrt{2}}{20} + \frac{4}{4\sqrt{2}}\cdot\left(\frac{1}{5}\right)\right]$$
$$= 2\left[\frac{-\sqrt{2}}{5} + \frac{1}{5\sqrt{2}}\right]$$
$$= 2\left[\frac{-\sqrt{2}}{5} + \frac{\sqrt{2}}{10}\right]$$
$$= \frac{-\sqrt{2}}{5}$$

$$2z + 2y = 48$$

5. $$z + y = 24$$
$$z = 24 - y$$

$$2z + x = 18$$
$$x = 18 - 2z$$
$$= 18 - 2(24 - y)$$
$$= 2y - 30$$

$$V = xyz$$
$$= (2y-30)y(24-y)$$
$$= 2y(y-15)(24-y)$$
$$= 2y(-y^2 + 39y - 360)$$
$$= -2y^3 + 78y^2 - 720y$$

$$0 = \frac{dV}{dy}$$
$$0 = -6y^2 + 15by - 720$$
$$y = 20 \text{ or } 6$$

$$\frac{d^2V}{dy^2} = -12y + 156$$

When $y = 20$, $\frac{d^V}{dy^2} = -84 < 0$

When $y = 6$, $\frac{d^V}{dy^2} = 84 > 0$

$\therefore V$ is max when $y = 20$

$$V = -2(20)^3 + 78(20)^2 - 720(20) = 800$$

6. (a) $\frac{dx}{d\theta} = 1 - \cos\theta, \quad \frac{dy}{d\theta} = \sin\theta$
$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$

Equation of tangent parallel to x-axis, $\frac{dy}{dx} = 0$,

$$\sin\theta = 0$$
$$\theta = \pi$$

$$y = 1 - \cos\pi$$
$$= 1 - (-1)$$
$$= 2$$

(b)
At $t = \frac{2\pi}{3}$,
$y = 1 - \cos\frac{2\pi}{3} = 1\frac{1}{2}$
$x = \frac{2\pi}{3} - \sin\frac{2\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$
$\frac{dy}{dx} = \frac{\sin\frac{2\pi}{3}}{1-\cos\frac{2\pi}{3}} = \frac{\sqrt{3}}{3}$

Equation of normal:

$$\frac{y - \frac{3}{2}}{x - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)} = -1 \div \frac{\sqrt{3}}{3}$$
$$= -\sqrt{3}$$
$$y - \frac{3}{2} = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3} - \frac{3}{2}$$
$$y = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$$

At P, $y = 0$,

$$0 = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$$
$$x = \frac{2\pi}{3}$$

At Q, $x = 0$

$$y = \frac{2\sqrt{3}\pi}{3}$$

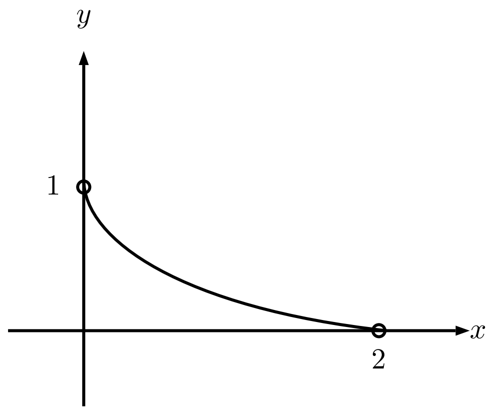
Area, $\frac{1}{2}(\frac{2\pi}{3})(\frac{2\sqrt{3}\pi}{3}) = \frac{2\sqrt{3}\pi^2}{9} units^2$

(c)

$\frac{d\theta}{dt} = 2$, $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$
Rate of change of $\frac{dy}{dx}$: $\frac{d(\frac{dy}{dx})}{dt}$
Note that $\frac{d(\frac{dy}{dx})}{dt} = \frac{d(\frac{dy}{dx})}{d\theta} \times \frac{d\theta}{dt}$
$\frac{d(\frac{dy}{dx})}{d\theta} = \frac{\cos\theta(1-\cos\theta) - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}$

At $\theta = \frac{\pi}{3}$, $\frac{d(\frac{dy}{dx})}{d\theta} = \frac{\frac{1}{2}(\frac{1}{2}) - \frac{3}{4}}{\frac{1}{4}}$=-2
$\therefore \frac{d(\frac{dy}{dx})}{dt} = -2 \times 2 = -4$units/s

7. (i)



(ii)

$$\frac{dx}{d\theta} = 4a\sin^3\theta\cos\theta$$
$$\frac{dy}{d\theta} = 4b\cos^3\theta(-\sin\theta)$$
$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$
$$= \frac{-4b\cos^3\theta\sin\theta}{4a\sin^3\theta\cos\theta}$$
$$= -\frac{b}{a}\frac{\cos^2\theta}{\sin^2\theta}$$
$$= -\frac{b}{a}\cot^2\theta$$

(iii)

Equation of tangent:

$y - b\cos^4\theta = -\frac{b}{a}\cot^2\theta(x - a\sin^4\theta)$

At P, $y = 0$,

$$-b\cos^4\theta = -\frac{b}{a}\cot^2\theta x + b\cot^2\theta\sin^4\theta$$
$$-b\cos^4\theta = -\frac{b}{a}\cot^2\theta x + b\cos^2\theta\sin^2\theta$$
$$-b\cos^2\theta = -\frac{b}{a}\frac{1}{\sin^2\theta}x + b\sin^2\theta$$
$$-b\cos^2\theta - b\sin^2\theta = -\frac{b}{a}\frac{1}{\sin^2\theta}x$$
$$-b = -\frac{b}{a}\frac{1}{\sin^2\theta}x$$
$$x = a\sin^2\theta$$
$$\therefore P(a\sin^2\theta, 0)$$

At Q, $x = 0$,

$$y - b\cos^4\theta = -\frac{b}{a}\cot^2\theta(0 - a\sin^4\theta)$$
$$y = b\cos^4\theta + b\cos^2\theta\sin^2\theta$$
$$= b\cos^2\theta(\cos^2\theta + \sin^2\theta)$$
$$= b\cos^2\theta$$
$$\therefore Q(0, b\cos^2\theta)$$

(iv)

$$\text{Area } \triangle OPQ = \frac{1}{2}(9\sin^2\theta)(b\cos^2\theta)$$
$$= \frac{ab}{2}\sin^2\theta\cos^2\theta$$
$$= \frac{ab}{2}(\sin\theta\cos\theta)^2$$
$$= \frac{ab}{2}\left(\frac{\sin 2\theta}{2}\right)^2$$
$$= \frac{ab}{8}\sin^2 2\theta$$

$$\frac{dA}{d\theta} = \frac{ab}{8}(2\sin 2\theta\cos 2\theta \cdot 2)$$
$$= \frac{ab}{2}(\sin 2\theta\cos 2\theta)$$
$$= \frac{ab}{4}\sin 4\theta$$

$$\frac{dA}{d\theta} = 0$$
$$\Rightarrow \sin 4\theta = 0$$
$$4\theta = 0, \pi, 2\pi$$
$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$$

For $0 < \theta < \frac{\pi}{2}$, $\quad \theta = \frac{\pi}{4}$.

$$\frac{d^2A}{d\theta^2} = \frac{ab}{4}(4\cos 4\theta)$$
$$= ab\cos 4\theta$$

When $\theta = \frac{\pi}{4}$, $\frac{d^2A}{d\theta^2} = ab\cos\pi = -ab < 0$,

since $a, b < 0$.

Hence A is max at $\theta = \frac{\pi}{4}$,

$$A = \frac{ab}{8}\left(\sin\frac{\pi}{2}\right)^2$$
$$= \frac{ab}{8}(1)^2$$
$$= \frac{ab}{8}$$

8. $y = e^{1+x^2}$

Differentiate w.r.t. x,

$\frac{dy}{dx} = 2xe^{1+x^2}$

Differentiate w.r.t. x,

$$\frac{d^2y}{dx^2} = 2(e^{1+x^2} + x(2x)e^{1+x^2})$$
$$= 2e^{1+x^2} + 4x^2e^{1+x^2}$$
$$= 2y + 2x\left(2xe^{1+x^2}\right)$$
$$= 2y + 2x\left(\frac{dy}{dx}\right)$$
$$= 2\left(y + x\frac{dy}{dx}\right) \text{ (shown)}$$

9.

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2}$$
$$(1+x^2)\frac{dy}{dx} = e^{\tan^{-1}x}$$

Differentiate w.r.t. x,

$$2x\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2} = \frac{1}{1+x^2}e^{\tan^{-1}x}$$
$$= \frac{dy}{dx}$$
$$(2x-1)\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2} = 0 \text{ (shown)}$$