1. At time t, $AB = 3t$, $AP = 500 - 4t$

   $\tan\theta = \frac{AB}{AP} = \frac{3t}{500-4t}$

   $\theta = \tan^{-1}\left(\frac{3t}{500-4t}\right)$ (shown)

   (i)

   $$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{3t}{500-4t}\right)^2} \times \frac{(500-4t)(3) - 3t(-4)}{(500-4t^2)}$$
   $$= \frac{(500-4t)^2}{(500-4t)^2 + (3t)^2} \times \frac{1500}{(500-4t)^2}$$
   $$= \frac{1500}{9t^2 + (500-4t)^2}$$
   $$= \frac{1500}{25t^2 - 4000t + 250000}$$
   $$= \frac{60}{t^2 - 160t + 10000}$$

   (ii)

   $\frac{d\theta}{dt} = \frac{60}{t^2-160t+10000} = \frac{60}{(t-80)^2+3600}$

   $\frac{d\theta}{dt}$ is maximum when $(t-80)^2 = 0$ i.e. when $t = 80$.

2. (i)

   sloped length $= \frac{3}{\sin\theta}$

   total area $= \left(8 - \frac{3}{\sin\theta}\right)(3) + \frac{1}{2}\left(\frac{3}{\tan\theta}\right)(3) = 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta$ (shown)

   (ii)

   $A = 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta$

   $\frac{dA}{d\theta} = 9\cot\theta\operatorname{cosec}\theta - \frac{9}{2}\operatorname{cosec}^2\theta = \frac{9}{2}\cdot\frac{2\cos\theta-1}{(\sin\theta)^2}$

   Let $\frac{dA}{d\theta} = 0$.

   $\implies 2\cos\theta = 1$.

   $\implies \theta = \frac{\pi}{3}$

   $A = 24 - 9\operatorname{cosec}\frac{\pi}{3} + \frac{9}{2}\cot\frac{\pi}{3} = 24 - \frac{18}{\sqrt{3}} + \frac{9}{2\sqrt{3}} = 24 - \frac{9\sqrt{3}}{2} = 16.2$ (3 s.f.)

θ	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{3}\right)$
$\frac{dA}{d\theta}$	$+$	0	$-$
	/	–	\

   The maximum area is 16.2 m^2.

3. $\frac{dx}{dt} = -2\sin t$

   $\frac{dy}{dt} = \cos t$

   $\frac{dy}{dx} = \frac{-\cos t}{2\sin t}$

   Equation of tangent at P, $t = \theta$:

$$
\begin{aligned}
y - \sin\theta &= \frac{-\cos\theta}{2\sin\theta}(x - 2\cos\theta) \\
(2\sin\theta)y - 2\sin^2\theta &= (-\cos\theta)x + 2\cos^2\theta \\
(2\sin\theta)y + (\cos\theta)x &= 2(\sin^2\theta + \cos^2\theta) \\
(\cos\theta)x + (2\sin\theta)y &= 2 \text{ (shown)}
\end{aligned}
$$

Equation of normal at P, $t = \theta$:

$$
\begin{aligned}
y - \sin\theta &= \frac{2\sin\theta}{\cos\theta}(x - 2cos\theta) \\
(\cos\theta)y - \sin\theta\cos\theta &= (2\sin\theta)x - 4\sin\theta\cos\theta \\
(2\sin\theta)x - (\cos\theta)y &= 3\sin\theta\cos\theta \text{ (shown)}
\end{aligned}
$$

(i)

Sub $x = 2\cos t, y = \sin t$ into equation of tangent at P

$$
\begin{aligned}
(\cos\theta)(2\cos t) + (2\sin\theta)(\sin t) &= 2 \\
\cos\theta\cos t + \sin\theta\sin t &= 1 \\
\cos(\theta - t) &= 1 \\
\theta - t &= 0 \\
t &= \theta
\end{aligned}
$$

Since the tangent at point P intersects the curve only once at $t = \theta$, the tangent does not cut the curve again.

(ii)

At A, $y = 0$

$$
\begin{aligned}
(2\sin\theta)x &= 3\sin\theta\cos\theta \\
x &= \frac{3}{2}\cos\theta
\end{aligned}
$$

At B, $x = 0$

$$
\begin{aligned}
-(\cos\theta)y &= 3\sin\theta\cos\theta \\
y &= -3\sin\theta
\end{aligned}
$$

mid point of AB : $\left(\frac{\frac{3}{2}\cos\theta + 0}{2}, \frac{-3\sin\theta + 0}{2}\right) = \left(\frac{3}{4}\cos\theta, -\frac{3}{2}\sin\theta\right)$

$x = \frac{3}{4}\cos\theta \implies \cos\theta = \frac{4}{3}x$

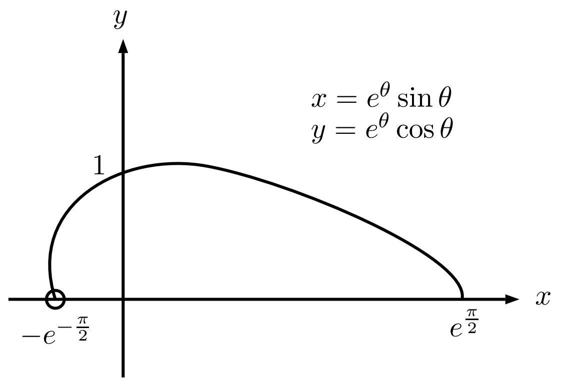
$y = -\frac{3}{2}\sin\theta \implies \sin\theta = \frac{-2}{3}y$

Cartesian equation of the locus of the mid-point of AB:

$$
\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
\frac{16x^2}{9} + \frac{4x^2}{9} &= 1 \\
\frac{x^2}{\left(\frac{3}{4}\right)^2} + \frac{y^2}{\left(\frac{3}{2}\right)^2} &= 1
\end{aligned}
$$

The locus is an ellipse, centre at (0, 0) with vertical major axis of length 3 units and horizontal minor axis of length $\frac{3}{2}$ units.

4. (i)



(ii)

$\frac{dx}{dt} = 0.1$, $\frac{dy}{dt} =?$ at $x = \frac{1}{2}e^{\frac{\pi}{6}}$

$e^{\theta}\sin\theta = \frac{1}{2}e^{\frac{\pi}{6}} \quad \Rightarrow \theta = \frac{\pi}{6}$

$\frac{dy}{dx} = \frac{e^{\theta}\cos\theta - e^{\theta}\sin\theta}{e^{\theta}\cos\theta + e^{\theta}\sin\theta} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$\begin{aligned}\frac{dy}{dt} &= \frac{\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)} \times 0.1 \\ &= \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{2}}(0.1) \\ &= \frac{\sqrt{3} - 1}{10(\sqrt{3} + 1)}\end{aligned}$$

(iii)

$\frac{dy}{dx} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

At point P, tangent // y-axis

$\Rightarrow \frac{dy}{dx}$ is undefined

$\Rightarrow \cos\theta + \sin\theta = 0$

$\Rightarrow \tan\theta = -1$

$\Rightarrow \theta = -\frac{\pi}{4} \left(\because -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}\right)$

$x = e^{-\frac{\pi}{4}}\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$

Equation of tangent at point P is $x = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$

(iv)

Coordinates of P: $\left(-\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}, \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}\right)$

$OP = \sqrt{\frac{1}{2}e^{-\frac{\pi}{2}} + \frac{1}{2}e^{-\frac{\pi}{2}}} = e^{-\frac{\pi}{4}}$

Note that P lies on the line $y = -x$ (since $\tan\theta = -1$) and $OP \perp OQ$, then Q lies on the line of $y = x$.

We now intersect $y = x$ with the curve C.

$e^{\theta}\sin\theta = e^{\theta}\cos\theta$

$\theta = \frac{\pi}{4}$. We sub this value into C to obtain coordinates of Q.

$x = e^{\frac{\pi}{4}} \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} e^{\frac{\pi}{4}}$

$y = e^{\frac{\pi}{4}} \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} e^{\frac{\pi}{4}}$

$OQ = \sqrt{\frac{1}{2}e^{\frac{\pi}{2}} + \frac{1}{2}e^{\frac{\pi}{2}}} = e^{\frac{\pi}{4}}$

Area of $\triangle POQ = \frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(e^{\frac{-\pi}{4}}\right)\left(e^{\frac{\pi}{4}}\right) = \frac{1}{2}$ units^2