Integration Application Revision: Solutions

1. (i)

$$\text{Required Area} = \int_{-\frac{2}{3}}^{0} \left(3 - \frac{12}{(3x+2)^2+4}\right)\ \mathrm{d}x$$
$$= \int_{-\frac{2}{3}}^{0} \left(3 - \frac{12}{(3x+2)^2+2^2}\right)\ \mathrm{d}x$$
$$= \int_{-\frac{2}{3}}^{0} \left(3 - 4\left[\frac{3}{(3x+2)^2+2^2}\right]\right)\ \mathrm{d}x$$
$$= \left[3x - \frac{4}{2}\tan^{-1}\left(\frac{3x+2}{2}\right)\right]_{-\frac{2}{3}}^{0}$$
$$= 0 - 2\left(\frac{\pi}{4}\right) - [-2-0]$$
$$= 2 - \frac{\pi}{2}$$

(ii)

$y = \frac{12}{(3x+2)^2+4}$

$\Rightarrow 3x = -2 \pm \sqrt{\frac{12}{y} - 4} = -2 \pm \sqrt{\frac{12-4y}{y}}$

$\Rightarrow x = -\frac{2}{3} \pm \frac{1}{3}\sqrt{\frac{12-4y}{y}}$

Since $x \geq -\frac{2}{3}$, we have $x = -\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}}$

$$\text{Required Volume} = \pi \int_{\frac{3}{2}}^{3} x^2\ \mathrm{d}y$$
$$= \pi \int_{\frac{3}{2}}^{3} \left(-\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}}\right)^2\ \mathrm{d}y$$
$$= 0.5125 \quad \text{(Using GC)}$$

2. (a)

$\frac{5x^2-2x+7}{(1-x)(2x^2+3)} = \frac{A}{1-x} + \frac{Bx+C}{2x^2+3}$, where

$A = \frac{5-2+7}{2+3} = 2$, (using cover-up rule)

$5x^2 - 2x + 7 = 2(2x^2+3) + (Bx+C)(1-x)$

Comparing the coefficients of x^2 and x^0, we have

$4 - B = 5 \quad \Rightarrow B = -1$, and

$6 + C = 7 \quad \Rightarrow C = 1$.

Hence we get $\frac{5x^2-2x+7}{(1-x)(2x^2+3)} = \frac{2}{1-x} + \frac{-x+1}{2x^2+3}$,

Therefore,

$$\int \frac{5x^2-2x+7}{(1-x)(2x^2+3)}\ \mathrm{d}x$$
$$= \int \frac{2}{1-x} + \frac{-x+1}{2x^2+3}\ \mathrm{d}x$$
$$= \int \frac{2}{1-x}\ \mathrm{d}x + \int \frac{-x+1}{2x^2+3}\ \mathrm{d}x$$
$$= \int \frac{2}{1-x}\ \mathrm{d}x - \int \frac{x}{2x^2+3}\ \mathrm{d}x + \int \frac{1}{2x^2+3}\ \mathrm{d}x$$
$$= -2\int \frac{-1}{1-x}\ \mathrm{d}x - \frac{1}{4}\int \frac{4x}{2x^2+3}\ \mathrm{d}x + \frac{1}{2}\int \frac{1}{x^2+\left(\sqrt{\frac{3}{2}}\right)^2}\ \mathrm{d}x$$
$$= -2\ln|1-x| - \frac{1}{4}\ln|2x^2+3| + \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{3}{2}}}\right) + c$$
$$= -2\ln|1-x| - \frac{1}{4}\ln|2x^2+3| + \frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{3}{2}}x\right) + c$$

b(i)

$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(e^{-x})] = -e^{-x}\cos(e^{-x})$

b(ii)

$$\int_0^n e^{-2x}\cos(e^{-x})\ \mathrm{d}x$$
$$= \int_0^n -e^{-x}(-e^{-x}\cos(e^{-x}))\ \mathrm{d}x$$
$$= [-e^{-x}(\sin(e^{-x}))]_0^n - \int_0^n (\sin(e^{-x}))(e^{-x})\ \mathrm{d}x$$
$$= [-e^{-n}(\sin(e^{-n}))] - [-e^0(\sin(e^0))] + \int_0^n -e^{-x}\sin(e^{-x})\ \mathrm{d}x$$
$$= -e^{-n}\sin(e^{-n}) + \sin 1 + [-\cos(e^{-x})]_0^n$$
$$= -e^{-n}\sin(e^{-n}) + \sin 1 + [-\cos(e^{-n}) + \cos(e^0)]$$
$$= -e^{-n}\sin(e^{-n}) - \cos(e^{-n}) + \sin 1 + \cos 1$$

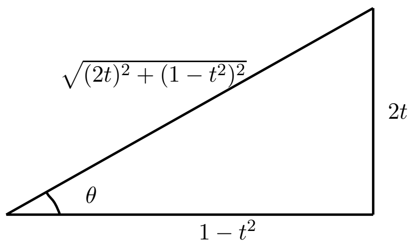
b(iii)

$$\int_0^\infty e^{-2x}\cos(e^{-x})\ \mathrm{d}x$$
$$= \lim_{n\to\infty}\int_0^n e^{-2x}\cos(e^{-x})\ \mathrm{d}x$$
$$= \lim_{n\to\infty}[-e^{-n}\sin(e^{-n}) - \cos(e^{-n}) + \sin 1 + \cos 1]$$
$$= 0 - \cos 0 + \sin 1 + \cos 1$$
$$= -1 + \sin 1 + \cos 1$$

3. a(i)

$t = \tan\frac{\theta}{2}$

$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$



Using the triangle above, the hypotenuse is
$\sqrt{(2t)^2+(1-t^2)^2}=\sqrt{4t^2+1-2t^2+t^4}=\sqrt{t^4+2t^2+1}=\sqrt{(t^2+1)^2}=t^2+1$
$\therefore \sin\theta=\frac{2t}{1+t^2}$ (shown)

a(ii)
$t=\tan\frac{\theta}{2}$
when $\theta=\frac{\pi}{2}: t=\tan\frac{\frac{\pi}{2}}{2}=1$
when $\theta=0: t=\tan\frac{0}{2}=0$
$\tan^{-1}t=\frac{\theta}{2}$
$\frac{1}{1+t^2}\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1}{2}$
$\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1+t^2}{2}$
$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\ \mathrm{d}\theta$$
$$=\int_0^1\frac{t+1}{\frac{2t}{1+t^2}+1}\left(\frac{2}{1+t^2}\ \mathrm{d}t\right)$$
$$=\int_0^1\frac{t+1}{\frac{2t+1+t^2}{1+t^2}}\left(\frac{2}{1+t^2}\ \mathrm{d}t\right)$$
$$=\int_0^1\frac{2(t+1)}{2t+1+t^2}\ \mathrm{d}t$$
$$=\int_0^1\frac{2(t+1)}{(t+1)^2}\ \mathrm{d}t$$
$$=\int_0^1\frac{2}{t+1}\ \mathrm{d}t$$
$$=2[\ln(t+1)]_0^1$$
$$=2\ln 2$$
(b)

$$\int e^{2v}\cos 3v \ \mathrm{d}v$$
$$= e^{2v}\left(\frac{1}{3}\sin 3v\right) - \int \frac{2}{3}e^{2v}\sin 3v \ \mathrm{d}v$$
$$= \frac{1}{3}e^{2v}\sin 3v - \frac{2}{3}\left[e^{2v}\left(-\frac{1}{3}\right)\cos(3v) - \int \frac{-2}{3}e^{2v}\cos(3v) \ \mathrm{d}v\right]$$
$$= \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v) - \int \frac{4}{9}e^{2v}\cos(3v) \ \mathrm{d}v$$
$$\therefore \frac{13}{9}\int e^{2v}\cos 3v \ \mathrm{d}v = \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v)$$
$$\int e^{2v}\cos 3v \ \mathrm{d}v = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

4. (a)

$$\int_0^b x|a-x| \ \mathrm{d}x = \int_0^a x(a-x) \ \mathrm{d}x + \int_a^b x(x-a) \ \mathrm{d}x$$
$$= \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a + \left[\frac{x^3}{3} - \frac{ax^2}{2}\right]_a^b$$
$$= \left(\frac{a^3}{2} - \frac{a^3}{3}\right) + \left(\frac{b^3}{3} - \frac{ab^2}{2}\right) - \left(\frac{a^3}{3} - \frac{a^3}{2}\right)$$
$$= \frac{a^3}{3} + \frac{b^3}{3} - \frac{ab^2}{2}$$

b(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3-x}{\sqrt{1-x}}\right) = \frac{-\sqrt{1-x} - (3-x)\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{1-x}}\right)}{1-x}$$
$$= \frac{-2+2x+3-x}{2(1-x)^{\frac{3}{2}}}$$
$$= \frac{x+1}{2(1-x)^{\frac{3}{2}}}$$

b(ii)

$$\int \frac{3-x}{x^2-3x+2} \ \mathrm{d}x$$
$$= \int \frac{3-x}{(x-2)(x-1)} \ \mathrm{d}x$$
$$= \int \left[\frac{1}{x-2} - \frac{2}{x-1}\right] \ \mathrm{d}x \quad \text{(by partial fractions)}$$
$$= \ln|x-2| - 2\ln|x-1| + c$$

b(iii)

$$\int \frac{1+x}{(1-x)^{\frac{3}{2}}} \tan^{-1}\sqrt{1-x}\ \mathrm{d}x$$
$$= \int \frac{1+x}{2(1-x)^{\frac{3}{2}}} \left(2\tan^{-1}\sqrt{1-x}\right)\ \mathrm{d}x$$
$$= 2\left(\frac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x} - \int 2\left(\frac{3-x}{\sqrt{1-x}}\right)\left(\frac{1}{1+(\sqrt{1-x})^2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{1-x}}\right)\ \mathrm{d}x$$
$$= 2\left(\frac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x} - \int \frac{3-x}{(2-x)(1-x)}\ \mathrm{d}x$$
$$= 2\left(\frac{3-x}{\sqrt{1-x}}\right)\tan^{-1}\sqrt{1-x} + \ln|x-2| - 2\ln|x-1| + c$$

5. (i)

When $x=0$, $t(2+t)=0 \quad \Rightarrow t=0$ or $t=-2$.

Coordinates are (0, 1) and (0, $\frac{1}{9}$).

(ii)

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2+2t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2}{(1-t)^3}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1+t)(1-t)^3}$

When tangent is parallel to y-axis, gradient is undefined, so we set denominator $=0$.

$(1+t)(1-t)^3 = 0 \quad \Rightarrow t=-1$ or $t=1$ (this corresponds to $x=3$ whic is the vertical asymptote)

Hence, equation of tangent is $x=-1$

(iii)

Let $u = 1-t$

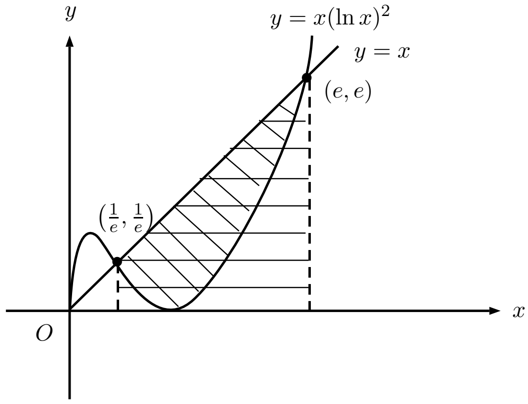
$\frac{\mathrm{d}u}{\mathrm{d}t} = -1$

When $t=0, u=1$

When $t=-2, u=3$

$$\text{Area} = -\int_{\frac{1}{9}}^{1} x\ \mathrm{d}y$$
$$= -\int_{-2}^{0} (2t+t^2)\cdot\frac{2}{(1-t)^3}\ \mathrm{d}t$$
$$= \int_{3}^{1} (2(1-u)+(1-u^2)\cdot\frac{2}{u^3}\ \mathrm{d}u$$
$$= -2\int_{1}^{3} \frac{u^2-4u+3}{u^3}\ \mathrm{d}u$$
$$= -2\int_{1}^{3} \left(\frac{1}{u} - \frac{4}{u^2} + \frac{3}{u^3}\right)\ \mathrm{d}u$$
$$= -2\left[\ln|u| + \frac{4}{u} - \frac{3}{2u^2}\right]_1^3$$
$$= -2\left[\left(\ln 3 + \frac{4}{3} - \frac{3}{18}\right) - \left(4 - \frac{3}{2}\right)\right] = \frac{8}{3} - 2\ln 3$$

6. (a)



Area of shaded region

$$= \frac{1}{2}\left(e+\frac{1}{e}\right)\left(e-\frac{1}{e}\right) - \int_{\frac{1}{e}}^{e} x(\ln x)^2 \ \mathrm{d}x \quad \text{(the term on the left is the area of trapezium)}$$

$$= \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) - \left(\left[\frac{x^2}{2}(\ln x)^2\right]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \frac{x^2}{2}\frac{2\ln x}{x} \ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) - \left(\left[\frac{e^2}{2}-\frac{1}{2e^2}\right] - \int_{\frac{1}{e}}^{e} x\ln x \ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) - \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) + \left(\left[\frac{x^2}{2}\ln x\right]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \frac{x^2}{2}\frac{1}{x} \ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(e^2+\frac{1}{e^2}\right) - \frac{1}{2}\left[\frac{x^2}{2}\right]_{\frac{1}{e}}^{e}$$

$$= \frac{1}{2}\left(e^2+\frac{1}{e^2}\right) - \frac{1}{2}\left(\frac{e^2}{2}-\frac{1}{2e^2}\right)$$

$$= \frac{1}{4}\left(e^2+\frac{3}{e^2}\right)$$

Area $= \frac{1}{4}\left(e^2+\frac{3}{e^2}\right) + \frac{1}{2}\left(e-\frac{1}{e}\right)^2$

(b)

Given $(x+2)^2 + 4(y-1)^2 = 4$

$(x+2)^2 = 4[1-(y-1)^2]$

$\Rightarrow x+2 = \pm 2\sqrt{1-(y-1)^2}$

The shaded region is bounded by the section of the ellipse where $x \leq 2$.

Hence $x = -2 - 2\sqrt{1-(y-1)^2}$

Volume of solid formed

$$= \pi \int_1^2 \left[(-4)^2 - x^2\right] \ \mathrm{d}y$$

$$= \pi \int_1^2 (-4)^2 - \left(-2-2\sqrt{1-(y-1)^2}\right)^2 \ \mathrm{d}y$$

$= 9.6$ units^3 (Using GC)

7. (i)

$$\int \frac{1}{1+(3-y)^2}\ \mathrm{d}y$$
$$= -\int \frac{-1}{1+(3-y)^2}\ \mathrm{d}y$$
$$= (-1)\tan^{-1}\left(\frac{3-y}{1}\right) + C$$
$$= -\tan^{-1}(3-y) + C$$

(ii)

$$\begin{aligned}
y &= 3 - \frac{x}{\sqrt{4-x^2}} \\
\frac{x}{\sqrt{4-x^2}} &= 3 - y \\
\frac{x^2}{4-x^2} &= (3-y)^2 \\
x^2 &= (3-y)^2(4-x^2) \\
x^2 &= 4(3-y)^2 - x^2(3-y)^2 \\
x^2 + x^2(3-y)^2 &= 4(3-y)^2 \\
x^2 &= \frac{4(3-y)^2}{1+(3-y)^2} \\
&= 4 - \frac{4}{1+(3-y)^2}
\end{aligned}$$

Volume of revolution about the y-axis

$$= \pi \int_1^3 x^2\ \mathrm{d}y$$
$$= \pi \int_1^3 4 - \frac{4}{1+(3-y)^2}\ \mathrm{d}y$$
$$= \pi \int_1^3 4\ \mathrm{d}y - 4\pi \int_1^3 \frac{1}{1+(3-y)^2}\ \mathrm{d}y$$
$$= \pi[4y]_1^3 - 4\pi[-\tan^{-1}(3-y)]_1^3$$
$$= 8\pi - 4\pi\tan^{-1}(2)$$

Using the substitution $x = 2\sin\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$

When $x = 1$, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

When $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

Area under the curve

$$= \int_0^1 3 - \frac{x}{\sqrt{4-x^2}}\ \mathrm{d}x$$
$$= \int_0^{\frac{\pi}{6}} \left(3 - \frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}\right)(2\cos\theta)\ \mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{6}} 6\cos\theta - \frac{4\sin\theta\cos\theta}{2\cos\theta}\ \mathrm{d}\theta$$
$$= [6\sin\theta + 2\cos\theta]_0^{\frac{\pi}{6}}$$
$$= 1 + \sqrt{3}$$