Vectors Revision 2: Solution

1. (i)

$$\mathbf{a} \times \mathbf{b} = 4\mathbf{a} \times \mathbf{c}$$
$$(\mathbf{a} \times \mathbf{b}) - (4\mathbf{a} \times \mathbf{c}) = \mathbf{0}$$
$$(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times 4\mathbf{c}) = \mathbf{0}$$
$$\mathbf{a} \times (\mathbf{b} - 4\mathbf{c}) = \mathbf{0}$$

$\therefore$ $\mathbf{a}$ is parallel to $\mathbf{b} - 4\mathbf{c}$.

$\mathbf{b} - 4\mathbf{c} = \alpha\mathbf{a}$

(ii)

$$\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \sqrt{126}$$
$$\frac{1}{2}|4\mathbf{a} \times \mathbf{c}| = \sqrt{126}$$
$$|\mathbf{a} \times \mathbf{c}| = \frac{\sqrt{126}}{2}$$
$$\left|\left(\frac{\mathbf{b} - 4\mathbf{c}}{\sqrt{3}}\right) \times \mathbf{c}\right| = \frac{\sqrt{126}}{2}$$
$$|\mathbf{b} \times \mathbf{c} - (4\mathbf{c} \times \mathbf{c})| = \frac{\sqrt{3}\sqrt{126}}{2}$$
$$|\mathbf{b} \times \mathbf{c}| = \frac{\sqrt{378}}{2}$$

(iii)

Area of parallelogram with adjacent sides OB and OC.

(iv)

$$(\mathbf{b} - 4\mathbf{c}) \cdot (\mathbf{b} - 4\mathbf{c}) = \sqrt{3}\mathbf{a} \cdot \sqrt{3}\mathbf{a}$$
$$|\mathbf{b}|^2 - 8\mathbf{b} \cdot \mathbf{c} + 16|\mathbf{c}|^2 = 3|\mathbf{a}|^2$$
$$\mathbf{b} \cdot \mathbf{c} = -\frac{10}{8}$$
$$\cos\theta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{-\frac{10}{8}}{1(2)}$$
$$\theta = 128.7^\circ$$

2. (i)

Convert line l into vector form:

$x = 5 - 3\lambda$
$y = 4\lambda$
$z = \lambda$

$$\therefore \mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

Since l lines on p_3, normal vector of p_3 will be perpendicular to l.

$$\begin{pmatrix}1\\1\\b\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$$

$$-3+4+b=0$$

$$b=-1 \text{ (shown)}$$

(ii)

Line $RS : r = \begin{pmatrix}-2\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}, \lambda \in \mathbb{R}$

Substitute this line to p_1,

$$\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$$

$$\lambda=\frac{8}{11}$$

$$\overrightarrow{OS}=\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}=\begin{pmatrix}-\frac{14}{11}\\\frac{52}{11}\\\frac{3}{11}\end{pmatrix}$$

(iii)

Note that $\left|\begin{pmatrix}2\\3\\-6\end{pmatrix}\right| = \sqrt{2^2+3^2+6^2}=\sqrt{49}=7$

Using the formula $\mathbf{r}\cdot\hat{\mathbf{n}}=d$,

$$p_1 : r\cdot\frac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\frac{10}{7}$$

$$p_2 : r\cdot\frac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\frac{-a}{7}$$

distance between two planes: $\frac{8}{7}=\left|\frac{10-(-a)}{7}\right|$

$\frac{8}{7}=\frac{10-(-a)}{7}$

$a=-2$

or

$\frac{8}{7}=\frac{(-a)-10}{7}$

$a=-18$

(iv)

$$p_1 : \begin{pmatrix}5\\2\\c\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$$

$$10+6-6c=10$$

$$c=1$$

(v)

Let point A be (5, 0, 0) which is a point on line l.

$$PF = \frac{\left|\overrightarrow{AP} \times \begin{pmatrix}-3\\4\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right|} = \frac{\left|\begin{pmatrix}0\\2\\1\end{pmatrix} \times \begin{pmatrix}-3\\4\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right|} = \frac{\left|\begin{pmatrix}-2\\-3\\6\end{pmatrix}\right|}{\left|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right|}$$

$$= \frac{7}{\sqrt{26}} = 1.3728$$

Since F is on p_1, perpendicular distance from F to p_2 is $\frac{8}{7}$.

Q is the reflection of F in p_2, hence $QF = 2\left(\frac{8}{7}\right) = \frac{16}{7}$.

Area $PFQ = \frac{1}{2}(1.3728\left(\frac{16}{7}\right) = 1.5689 = 1.57$

3. (i)

$$\begin{aligned}(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) &= |\mathbf{b}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b}\\ &= |\mathbf{b}|^2 + 9|\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos 60^\circ\\ |\mathbf{b}-\mathbf{a}|^2 &= 10|\mathbf{b}|^2 - 2(3|\mathbf{b}|)|\mathbf{b}|\frac{1}{2}\\ |\mathbf{b}-\mathbf{a}| &= \sqrt{7}|\mathbf{b}|\end{aligned}$$

Therefore, $k = \sqrt{7}$.

(ii)

$\mathbf{c} = \frac{1}{3}\mathbf{a}$

$\overrightarrow{CA} = \frac{2}{3}\mathbf{a}$

$$\begin{aligned}\text{Shortest distance of } C \text{ to } l &= \left|\overrightarrow{CA} \times (\mathbf{b} \hat{-} \mathbf{a})\right|\\ &= \frac{\left|\frac{2}{3}\mathbf{a} \times (\mathbf{b}-\mathbf{a})\right|}{|\mathbf{b}-\mathbf{a}|}\\ &= \frac{\left|\frac{2}{3}\mathbf{a}\times\mathbf{b} - \frac{2}{3}\mathbf{a}\times\mathbf{a}\right|}{|\mathbf{b}-\mathbf{a}|}\\ &= \frac{2|\mathbf{a}\times\mathbf{b}|}{3|\mathbf{b}-\mathbf{a}|} \quad \because \mathbf{a}\times\mathbf{a} = 0\\ &= \frac{2|\mathbf{a}||\mathbf{b}|\sin 60^\circ}{3|\mathbf{b}-\mathbf{a}|}\\ &= \frac{6|\mathbf{b}|^2\frac{\sqrt{3}}{2}}{3\sqrt{7}|\mathbf{b}|}\\ &= \frac{\sqrt{3}|\mathbf{b}|}{\sqrt{7}}\\ &= \sqrt{\frac{3}{7}}|\mathbf{b}|\end{aligned}$$

4. $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) = \frac{1}{2}(2\mathbf{a}+\mathbf{b})$

$\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OM} = \frac{2}{3} \times \frac{1}{2}(2\mathbf{a}+\mathbf{b}) = \frac{1}{3}(2\mathbf{a}+\mathbf{b})$

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{1}{3}(2\mathbf{a} + \mathbf{b}) - \mathbf{a} = \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}\overrightarrow{AB}$

Since $\overrightarrow{AN}$ is parallel to $\overrightarrow{AB}$ and A is the common point, hence, A, B and N are collinear.

Since P is on AB, $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, where $\lambda \in \mathbb{R}$

$$\overrightarrow{MP} \cdot \overrightarrow{AB} = 0$$

$$\left[\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) - \frac{1}{2}(2\mathbf{a} + \mathbf{b})\right] \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\left[\left(\lambda - \frac{1}{2}\right)\mathbf{b} - \lambda\mathbf{a}\right] \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\left(\lambda - \frac{1}{2}\right)|\mathbf{b}|^2 - \left(\lambda - \frac{1}{2}\right)\mathbf{a} \cdot \mathbf{b} - \lambda\mathbf{a} \cdot \mathbf{b} + \lambda|\mathbf{a}|^2 = 0$$

Since $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos AOB = 2 \times 3 \cos 60° = 3$,

$$\text{Hence },9\left(\lambda - \frac{1}{2}\right) - 3\left(\lambda - \frac{1}{2}\right) - 3\lambda + 4\lambda = 0$$

$$\lambda = \frac{3}{7}$$

$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b} - \mathbf{a}) = \frac{1}{7}(4\mathbf{a} + 3\mathbf{b})$

5. (i)

$$l_1 : r = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since $(1, 0, 1)$ is on l_1 and p_1,

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = a$$

$$1 + 0 + 1 = 2$$

$$a = 2 \text{ (shown)}$$

(ii)

Let N be the foot of perpendicular from A to p_1.

$$l_{AN} : r = \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}$$

Let $\overrightarrow{ON} = \begin{pmatrix} 18 + \alpha \\ 2 + 3\alpha \\ \alpha \end{pmatrix}$ for some value of α.

Since N is a point on p_1.

$$\begin{pmatrix} 18+\alpha \\ 2+3\alpha \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 2$$

$$\begin{aligned} 18+\alpha+6+9\alpha+\alpha &= 2 \\ 24+11\alpha &= 2 \\ 11\alpha &= -22 \\ \alpha &= -2 \end{aligned}$$

$$\overrightarrow{ON} = \begin{pmatrix} 18-2 \\ 2-6 \\ -2 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ -2 \end{pmatrix} \quad \therefore N(16,-4,-2)$$

(iii)

Since B is on l_1,

$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix} - \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -17+2\lambda \\ -2+\lambda \\ 1-5\lambda \end{pmatrix}$$

$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{(-17+2\lambda)^2+(-2+\lambda)^2+(1-5\lambda)^2} \\ |\overrightarrow{AB}| &= \sqrt{(289-68\lambda+4\lambda^2)+(4-4\lambda+\lambda^2)+(1-10\lambda+25\lambda^2)} \\ |\overrightarrow{AB}| &= \sqrt{294-72\lambda+30\lambda^2} \\ |\overrightarrow{AB}|^2 &= 294-72\lambda+30\lambda^2 \end{aligned}$$

For shortest distance from A to l_1,

$|\overrightarrow{AB}|^2 = 30\lambda^2 - 72\lambda + 294$ must be minimum

We differentiate $30\lambda^2 - 72\lambda + 294$ w.r.t λ and set it to be zero,

$$\begin{aligned} 60\lambda - 72 &= 0 \\ \lambda &= \frac{6}{5} \end{aligned}$$

$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ \frac{6}{5} \\ 7 \end{pmatrix} \text{ or } \frac{1}{5}\begin{pmatrix} 17 \\ 6 \\ 35 \end{pmatrix}$$

(iv)

$$\text{direction vector of } l_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ -(1-2) \\ b \end{pmatrix} = \begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}$$

To find a common point between p_2 and p_3 by letting $y=0$: (you can set either of x, y or z to be any value you want)

$x + z = 1 - - - - (1)$

$2x + z = 4 - - - - (2)$

Solve (1) and (2):

$x = 3, z = -2$

Hence $l_2 : r = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}, \mu \in \mathbb{R}$ (shown)

6. a(i)

Since l lines on p_1, l is perpendicular to the normal of p_1. $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ \lambda \\ 4 \end{pmatrix} = 0$

$\lambda = 10$

Since l lines on p_2, (1,1,1) is a point on p_2.

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \\ 4 \end{pmatrix} = \mu$

$\mu = 17$

(ii)

Let θ be the angle between p_1 and p_2.

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \\ 4 \end{pmatrix}}{\sqrt{1^2 + 4^2 + 2^2}\sqrt{3^2 + 10^2 + 4^2}}$$

$$\cos\theta = \frac{51}{\sqrt{21}\sqrt{125}}$$

$$\theta = 5.4869^\circ$$

Acute angle between p_1 and $p_3 = 2(5.4869^\circ) = 11.0^\circ$

b(i)

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$\begin{aligned}\overrightarrow{AC} &= \mathbf{c} - \mathbf{a} \\ &= \lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{a} \\ &= (\lambda - 1)\mathbf{a} + \mu\mathbf{b} \\ &= -\mu\mathbf{a} + \mu\mathbf{b} \\ &= \mu(\mathbf{b} - \mathbf{a})\end{aligned}$$

Since $\overrightarrow{AC} = \mu\overrightarrow{AB}$ for some $\mu \in \mathbb{R} \setminus \{0\}$, A, B, C are collinear.

(ii)

$|\mathbf{c} \times \mathbf{a}|(\mathbf{b} - \mathbf{a}) = (\mathbf{c} \cdot \mathbf{d})\mathbf{d} \Rightarrow \overrightarrow{AB} = k\overrightarrow{OD}$ for some $k \in \mathbb{R}$, as $|\mathbf{c} \times \mathbf{a}| \neq 0$ since O is not on $\overrightarrow{AC}$.

This implies that $\overrightarrow{AB}$ is parallel to $\overrightarrow{OD}$.

Since $\overrightarrow{AB} = \mu\overrightarrow{AC}$ for some $\mu \in \mathbb{R}$, so $\overrightarrow{AC}$ is parallel to $\overrightarrow{OD}$.