Vectors Revision 1: Solution

1. (a)

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}\alpha\\7\\2\end{pmatrix}=\alpha+14+6=\alpha+20$$

Hence, A lies in P_1.

Since A also lies in P_2,

$$7=\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}3\\5\\\beta\end{pmatrix}=3+10+3\beta=13+3\beta$$

$$\begin{aligned}13+3\beta&=7\\3\beta&=-6\\\beta&=-2\end{aligned}$$

(b)

$$\mathbf{m}=\begin{pmatrix}\alpha\\7\\2\end{pmatrix}\times\begin{pmatrix}3\\5\\-2\end{pmatrix}=\begin{pmatrix}-24\\2\alpha+6\\5\alpha-21\end{pmatrix}$$

$$l:\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-24\\2\alpha+6\\5\alpha-21\end{pmatrix},\lambda\in\mathbb{R}.$$

Intersecting the two lines together,

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-24\\2\alpha+6\\5\alpha-21\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+k\begin{pmatrix}4\\-2\\1\end{pmatrix}$$

$$\lambda\begin{pmatrix}-24\\2\alpha+6\\5\alpha-21\end{pmatrix}=k\begin{pmatrix}4\\-2\\1\end{pmatrix}$$

Comparing **i**-component,

$$\begin{aligned}-24\lambda&=4k\\k&=-6\lambda\end{aligned}$$

Comparing **j**-component,

$$\begin{aligned}\lambda(2\alpha+6)&=-2k\\\lambda(2\alpha+6)&=-2(-6\lambda)\\2\alpha+6&=12\\2\alpha&=6\\\alpha&=3\text{ (shown)}\end{aligned}$$

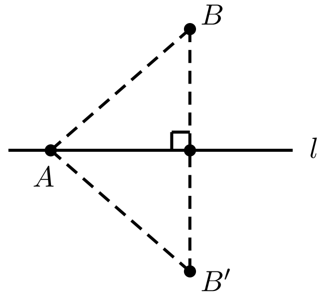
(c)

$$\theta = \cos^{-1} \frac{\left| \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} \right|}{\sqrt{3^2+7^2+2^2}\sqrt{3^2+5^2+2^2}}$$
$$= \cos^{-1} \left| \frac{9+35-4}{\sqrt{62}\sqrt{38}} \right|$$
$$= 34.5^\circ$$

(d)

Let F be the foot of perpendicular.

$$\overrightarrow{BF} \cdot \mathbf{m} = 0$$
$$(\overrightarrow{OF} - \overrightarrow{OB}) \cdot \mathbf{m} = 0$$
$$\left(\begin{pmatrix} 1+4k \\ 2-2k \\ 3+k \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} \right) \cdot \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = 0$$
$$\begin{pmatrix} -1+4k \\ 6-2k \\ -5+k \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = 0$$
$$-4+16k-12+4k-5+k = 0$$
$$21k = 21$$
$$k = 1$$
$$\therefore \overrightarrow{OF} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$$



Using midpoint thereom,

$$\overrightarrow{OF} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$$
$$\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB}$$
$$= 2 \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AB'} = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}.$$

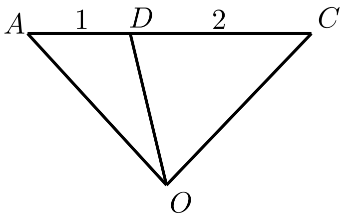
reflected line $l: r = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\overrightarrow{AB'}$

$= \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}7\\2\\-3\end{pmatrix}, \lambda \in \mathbb{R}.$

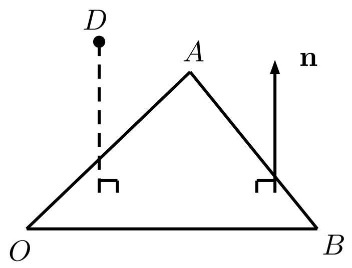
2. For any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ is the vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$. Hence,

$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0$

(a)



$\overrightarrow{OD} = \frac{2\mathbf{a}+\mathbf{c}}{3} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}$



If n is the normal vector of the plane OAB, note that $\mathbf{a}\times\mathbf{b}$ is parallel to $\mathbf{n}$.

Hence, $\mathbf{a}\times\mathbf{b}$ is the direction vector of l.

$l: \mathbf{r} = \overrightarrow{OD} + \lambda(\mathbf{a}\times\mathbf{b})$

$= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} + \lambda(\mathbf{a}\times\mathbf{b}), \lambda \in \mathbb{R}.$

(b)

Using $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, we first find the equation of the plane p.

$p: \mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$ (by first part)

Sub l into p to find the point of intersection between the line and plane,

$$\left[\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} + \lambda(\mathbf{a}\times\mathbf{b})\right]\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

$$\frac{2}{3}\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) + \frac{1}{3}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) + \lambda(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

$$\frac{1}{3}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) + \lambda|\mathbf{a}\times\mathbf{b}|^2 = 0$$

$$\lambda = \frac{-\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{3|\mathbf{a}\times\mathbf{b}|^2} \text{ (Shown).}$$

3. i. $\begin{pmatrix}3\\5\\2\end{pmatrix}\cdot\begin{pmatrix}13\\-5\\-7\end{pmatrix} = 39 - 25 - 14 = 0$

For l_1,

$x = 3\lambda + 11$

$y = 5\lambda + 7$

$z = 2\lambda + 4$

$$l_1 : \mathbf{r} = \begin{pmatrix} 11 \\ 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}.$$

Since $\mathbf{n}_\pi \cdot \mathbf{m}_{\mathbf{l_1}} = 0$, the line l_1 is parallel to π.

ii.

$$\cos\theta = \frac{\left| \begin{pmatrix} 11 \\ 7 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -5 \\ -7 \end{pmatrix} \right|}{\sqrt{11^2 + 7^2 + 4^2}\sqrt{13^2 + 5^2 + 7^2}}$$

$$= \frac{80}{\sqrt{186}\sqrt{243}}$$

$$\theta = 67.9^\circ$$

Required angle $= 90^\circ - 67.9^\circ = 22.1^\circ$

iii. P is a point on l_1.

$$\text{Required distance} = \left| \overrightarrow{OP} \times \hat{\mathbf{m}}_{\mathbf{1}} \right|$$

$$= \left| \begin{pmatrix} 11 \\ 7 \\ 4 \end{pmatrix} \times \frac{1}{\sqrt{3^2 + 5^2 + 2^2}} \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{38}} \left| \begin{pmatrix} -6 \\ -10 \\ 34 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{38}} \left(\sqrt{6^2 + 10^2 + 34^2} \right)$$

$$= 5.83$$

iv. $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \times \mathbf{d} = \mathbf{0}$ implies that $\mathbf{d}$ is parallel to $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$.

$\therefore$ l_1 and l_2 are parallel lines.

4. (a) i. By ratio theorem,

$$\overrightarrow{OR} = \frac{2\overrightarrow{OP} + 3\overrightarrow{OQ}}{5}$$

$$= \frac{2}{5}\mathbf{p} + \frac{3}{5}\mathbf{q}$$

ii.

$$\begin{aligned}
\text{required length} &= \left|\overrightarrow{OR}\cdot\hat{\overrightarrow{OP}}\right| \\
&= \left|\left(\frac{2}{5}\mathbf{p}+\frac{3}{5}\mathbf{q}\right)\cdot\hat{\mathbf{p}}\right| \\
&= \left|\left(\frac{2}{5}\mathbf{p}+\frac{3}{5}\mathbf{q}\right)\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\right| \\
&= \frac{1}{|\mathbf{p}|}\left|\frac{2}{5}\mathbf{p}\cdot\mathbf{p}+\frac{3}{5}\mathbf{q}\cdot\mathbf{p}\right| \\
&= \frac{1}{|\mathbf{p}|}\left|\frac{2}{5}|\mathbf{p}|^2+\frac{3}{5}\mathbf{p}\cdot\mathbf{q}\right| \\
&= \frac{\left|2|\mathbf{p}|^2+3\mathbf{p}\cdot\mathbf{q}\right|}{5|\mathbf{p}|}
\end{aligned}$$

(b) i.

$$\begin{aligned}
l_2: \quad x &= 4+\mu \\
y &= -1+\mu \\
z &= -3+2\mu
\end{aligned}$$

$$l_2: \mathbf{r} = \begin{pmatrix}4\\-1\\-3\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}, \mu\in\mathbb{R}.$$

Since l_1 and l_2 intersect,

$$\begin{pmatrix}1\\0\\-3\end{pmatrix} + \lambda\begin{pmatrix}k\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\-3\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\-3\end{pmatrix} - \begin{pmatrix}4\\-1\\-3\end{pmatrix} + \lambda\begin{pmatrix}k\\-1\\1\end{pmatrix} = \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}-3\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}k\\-1\\1\end{pmatrix} = \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\lambda\begin{pmatrix}k\\-1\\1\end{pmatrix} - \mu\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\0\end{pmatrix}$$

$$\begin{aligned}
\therefore k\lambda-\mu &= 3 \\
-\lambda-\mu &= -1 \quad \ldots(1) \\
\lambda-2\mu &= 0 \quad \ldots(2)
\end{aligned}$$

Solving for (1) and (2), $\mu = \frac{1}{3}, \lambda = \frac{2}{3}$.

$$\begin{aligned}
k\left(\frac{2}{3}\right) - \frac{1}{3} &= 3 \\
\frac{2}{3}k &= \frac{10}{3} \\
k &= 5 \text{ (Shown)}
\end{aligned}$$

ii. Let $\mathbf{n}$ be the normal vector of π.

$$\begin{aligned}\mathbf{n} &= \mathbf{m_1} \times \mathbf{m_2}\\ &= \begin{pmatrix}5\\-1\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\2\end{pmatrix}\\ &= \begin{pmatrix}-3\\-9\\6\end{pmatrix} = 3\begin{pmatrix}-1\\-3\\2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{r}\cdot\mathbf{n} &= \mathbf{a}\cdot\mathbf{n}\\ \mathbf{r}\cdot\begin{pmatrix}-1\\-3\\2\end{pmatrix} &= \begin{pmatrix}1\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\-3\\2\end{pmatrix} = -7\\ \therefore \mathbf{r}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix} &= 7\end{aligned}$$

iii. Let B be the point (1,0,-3) from l_1. So B lies on π.

$$\begin{aligned}\text{required distance} &= \left|\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right|\\ &= \left|\left[\begin{pmatrix}1\\0\\-3\end{pmatrix} - \begin{pmatrix}-7\\1\\-2\end{pmatrix}\right]\cdot\frac{1}{\sqrt{1+3^2+2^2}}\begin{pmatrix}1\\3\\-2\end{pmatrix}\right|\\ &= \frac{1}{\sqrt{14}}\left|\begin{pmatrix}8\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}\cdot\right|\\ &= \frac{1}{\sqrt{14}}\left|8-3+2\right|\\ &= \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}\end{aligned}$$

5. i. Since P_2 is perpendicular to P_3, $\mathbf{n_2}$ is also perpendicular to $\mathbf{n_3}$.

$$\begin{aligned}\mathbf{n_2}\cdot\mathbf{n_3} &= 0\\ \begin{pmatrix}-1\\\mu\\5\end{pmatrix}\cdot\begin{pmatrix}-3\\-4\\1\end{pmatrix} &= 0\\ 3-4\mu+5 &= 0\\ -4\mu &= -8\\ \mu &= 2\end{aligned}$$

ii. Using GC to solve

$x - 4y - 13z = 3,$

$-x + 2y + 5z = -5,$

we get $\mathbf{r} = \begin{pmatrix}7\\1\\0\end{pmatrix} + \alpha\begin{pmatrix}-3\\-4\\1\end{pmatrix}, \alpha \in \mathbb{R}.$

iii. Since F is on l, $\overrightarrow{OF} = \begin{pmatrix}7\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}$ for some fixed $\lambda \in \mathbb{R}$.

$$\overrightarrow{AF} \cdot \mathbf{m} = 0$$

$$(\overrightarrow{OF} - \overrightarrow{OA}) \cdot \mathbf{m} = 0$$

$$\left[\begin{pmatrix} 7-3\lambda \\ 1-4\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 6-3\lambda \\ 2-4\lambda \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = 0$$

$$-18 + 9\lambda - 8 + 16\lambda + \lambda = 0$$

$$26\lambda = 26$$

$$\lambda = 1$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

Coordinate of F is $(4, -3, 1)$.

iv. Note that $C(3, 0, 0)$ is a point on P_1.

$$\begin{aligned} \text{required distance} &= \left|\overrightarrow{AC} \cdot \mathbf{n_1}\right| \\ &= \left|\left[\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right] \cdot \frac{1}{\sqrt{1^2+4^2+13^2}} \begin{pmatrix} 1 \\ -4 \\ -13 \end{pmatrix}\right| \\ &= \frac{1}{\sqrt{186}} \left|\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ -13 \end{pmatrix}\right| \\ &= \frac{|2-4|}{\sqrt{186}} = \frac{2}{\sqrt{186}} \end{aligned}$$

6. i.

$$\begin{aligned} l: \quad \mathbf{r} &= \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + \lambda \left(\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \text{required angle :} \quad \cos\theta &= \frac{\left|\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right|}{\sqrt{2^2+2^2}\sqrt{1+2^2+1}} \\ &= \frac{|6|}{\sqrt{8}\sqrt{6}} \\ \theta &= 30^\circ \end{aligned}$$

ii.

$$\begin{aligned} \text{Shortest distance} &= |\overrightarrow{OA}| \sin 30^\circ \\ &= \sqrt{2^2+2^2} \sin 30^\circ \\ &= \sqrt{8} \sin 30^\circ \\ &= \sqrt{2} \end{aligned}$$

iii. Substitute l into π,

$$\begin{pmatrix} 0-\lambda \\ -2-2\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= -2\lambda + 2 + 2\lambda + 0$$

$$=2$$

Hence, l is in π.

iv. π_2 contains $\mathbf{m_l}$ and $\mathbf{n_1}$ as direction vectors.

$$\mathbf{n_2} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = -4 + 10 = 6$$

$$\therefore x + 2y + 5z = 6$$

v.

$$\mathbf{n_3} = \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\hat{\mathbf{n_3}} = \frac{1}{\sqrt{1+2^2+1}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Since π_3 is at a distance of 6 from O,

$$\mathbf{r} \cdot \hat{\mathbf{n_3}} = \pm\sqrt{6}$$

$$\mathbf{r} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \pm\sqrt{6}$$

$$\mathbf{r} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \pm 6$$

7. (a) $\mathbf{r} \cdot (\mathbf{r} - \mathbf{a}) = 0 \quad \Longrightarrow \overrightarrow{OP} \cdot \overrightarrow{PA} = 0$. Hence, $\angle OPA = 90°$. Since O and A are fixed points, the locus of P is a circle with OA as diameter.

(b) i. Since $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$ for any $k \in \mathbb{R}$, l_1 is not parallel to l_2.

Equate l_1 to l_2:

$$\begin{pmatrix}0\\7\\6\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}=\begin{pmatrix}-3\\6\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}$$
$$\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}-\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}=\begin{pmatrix}-3\\6\\-4\end{pmatrix}-\begin{pmatrix}0\\7\\6\end{pmatrix}$$

$$\begin{aligned}-3\lambda-2\mu&=-3\\2\lambda+5\mu&=-1\\2\lambda-6\mu&=-10\end{aligned}$$

Using GC to solve these simultaneous equations, there is no solution.

Hence l_1 and l_2 are non-parallel and non-intersecting, so they are skew lines.

ii. Let $\mathbf{n}$ be a vector that is perpendicular to l_1 and l_2.

$$\begin{aligned}\mathbf{n}&=\mathbf{m_1}\times\mathbf{m_2}\\&=\begin{pmatrix}-3\\2\\2\end{pmatrix}\times\begin{pmatrix}2\\-5\\6\end{pmatrix}\\&=\begin{pmatrix}22\\22\\11\end{pmatrix}=11\begin{pmatrix}2\\2\\1\end{pmatrix}\end{aligned}$$

Let $P_1=\begin{pmatrix}0\\7\\6\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}$ and $P_2=\begin{pmatrix}-3\\6\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}$ for some fixed $\lambda,\mu\in\mathbb{R}$.

$$\begin{aligned}\overrightarrow{P_1P_2}&=\overrightarrow{OP_2}-\overrightarrow{OP_1}\\&=\begin{pmatrix}-3\\6\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}-\left[\begin{pmatrix}0\\7\\6\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}\right]\\&=\begin{pmatrix}-3\\-1\\-10\end{pmatrix}+\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}-\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}\end{aligned}$$

Since $\overrightarrow{P_1P_2}$ and $\mathbf{n}$ are both perpendicular to l_1 and l_2, $\overrightarrow{P_1}P_2$ is parallel to $\mathbf{n}$.

$$\overrightarrow{P_1P_2}=k\mathbf{n}\qquad\text{for some } k\in\mathbb{R}$$

$$\begin{pmatrix}-3\\-1\\-10\end{pmatrix}+\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}-\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}=k\begin{pmatrix}2\\2\\1\end{pmatrix}$$
$$\mu\begin{pmatrix}2\\-5\\6\end{pmatrix}-\lambda\begin{pmatrix}-3\\2\\2\end{pmatrix}-k\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}3\\1\\10\end{pmatrix}$$

Solving system of linear equations for μ,λ,k, we get $k=-2,\lambda=-1,\mu=1$.

$$\therefore|\overrightarrow{P_1P_2}|=\left|\begin{pmatrix}-4\\-4\\-2\end{pmatrix}\right|=\sqrt{4^2+4^2+2^2}=6$$

8. (i)

$\overrightarrow{AC}=12\mathbf{i}+\mathbf{j}$

Equation of line AC : $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}$

Cartesian equation of line is $\frac{x}{12} = y, z = 5$.

(ii)

Let the acute angle between AC and RC be x.

$\cos x = \frac{\left| \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \right|}{\sqrt{145}\sqrt{26}}$

Therefore $x = 89.1°$

Shortest distance $= |\overrightarrow{CR}| \sin 89.07° = \sqrt{26} \sin 89.07° = 5.10$

(iii)

$\overrightarrow{OC} = \frac{\overrightarrow{OT} + (\lambda - 1)\overrightarrow{OA}}{\lambda}$

$\overrightarrow{OT} = \lambda\overrightarrow{OC} - (\lambda - 1)\overrightarrow{OA}$

$= \lambda \begin{pmatrix} 12 \\ 1 \\ 5 \end{pmatrix} - (\lambda - 1) \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 12\lambda \\ \lambda \\ 5 \end{pmatrix}$

Area of triangle $ORT = \frac{1}{2} \left| \overrightarrow{OT} \times \overrightarrow{OR} \right|$

Area of triangle OVM

$= \frac{1}{2} \left| \frac{\overrightarrow{OT}}{|\overrightarrow{OT}|} \times \frac{\overrightarrow{OR}}{2} \right|$

$= \frac{1}{2\sqrt{25 + 145\lambda^2}} \times \frac{1}{2} \left| \overrightarrow{OT} \times \overrightarrow{OR} \right|$

Therefore the ratio of triangle OVM to area of triangle ORT is $1 : 2\sqrt{25 + 145\lambda^2}$.