JC Revision: Vectors 2 Solutions

1. (i)

   $\mathbf{a}$ and $2\mathbf{a}+\mathbf{b}$ are perpendicular

   $$\begin{aligned}\Rightarrow \mathbf{a}\cdot(2\mathbf{a}+\mathbf{b}) &= 0\\ 2\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b} &= 0\\ 2|\mathbf{a}|^2+\mathbf{a}\cdot\mathbf{b} &= 0\\ \mathbf{a}\cdot\mathbf{b} &= -2|\mathbf{a}|^2\\ \mathbf{a}\cdot\mathbf{b} &= -8\\ |\mathbf{a}||\mathbf{b}|\cos\left(\frac{3\pi}{4}\right) &= -8\\ 2|\mathbf{b}|\left(-\frac{1}{\sqrt{2}}\right) &= -8\\ |\mathbf{b}| &= 4\sqrt{2}\end{aligned}$$

   (ii)

   Length of projection of $\mathbf{a}$ onto $\mathbf{b}$

   $$\begin{aligned}&= |\mathbf{a}\cdot\hat{\mathbf{b}}|\\ &= \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right|\\ &= \left|\frac{-8}{4\sqrt{2}}\right| = \sqrt{2}\end{aligned}$$

   (iii)

   $\overrightarrow{PB} = \frac{2}{5}\mathbf{b}$

   Area of triangle APB

   $$\begin{aligned}&= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{PB}|\\ &= \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times\frac{2}{5}\mathbf{b}\right|\\ &= \frac{1}{2}\left|\frac{2}{5}\mathbf{b}\times\mathbf{b}-\frac{2}{5}\mathbf{a}\times\mathbf{b}\right|\\ &= \frac{1}{2}\left|\mathbf{0}-\frac{2}{5}\mathbf{a}\times\mathbf{b}\right|\\ &= \frac{1}{5}|\mathbf{a}\times\mathbf{b}|\\ &= \frac{1}{5}\left||\mathbf{a}||\mathbf{b}|\sin\left(\frac{3\pi}{4}\right)\right|\\ &= \frac{1}{5}(2)(4\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{8}{5}\end{aligned}$$

2. (i)

   $\mathbf{a}\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$

   $$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}-1\\4+1\\2\end{pmatrix} = \begin{pmatrix}-1\\5\\2\end{pmatrix} = -\mathbf{i}+5\mathbf{j}+2\mathbf{k}$$

   $|\mathbf{a}\times\mathbf{b}|$ is the area of the parallelogram with two adjacent sides OA and OB or twice the area of $\triangle OAB$.

   (ii)

   Area of $\triangle OAB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2}\sqrt{1^2+5^2+2^2} = \frac{1}{2}\sqrt{30}$ sq units

   (iii)

   $$\mathbf{a}-\mathbf{b} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$$

   Equation of the line through the points A and B:

   $$\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} \text{ where } \lambda\in\mathbb{R}$$

   or

   $$\mathbf{r} = \begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} \text{ where } \lambda\in\mathbb{R}$$

   (iv)

   $$\overrightarrow{OC} = \begin{pmatrix}-13\\2\\3\end{pmatrix}$$

   $\overrightarrow{OM} = \begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}2+\lambda\\-\lambda\\1+3\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

   $$\overrightarrow{CM} = \overrightarrow{OM}-\overrightarrow{OC} = \begin{pmatrix}15+\lambda\\-\lambda-2\\3\lambda-2\end{pmatrix}$$

   Since $\overrightarrow{CM}$ is perpendicular to l,

   $$\overrightarrow{CM}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix} = 0$$

   $15+\lambda+\lambda+2+9\lambda-6=0$

   $11\lambda=-11$

   $\lambda=-1$

   $$\therefore \overrightarrow{OM} = \begin{pmatrix}1\\1\\-2\end{pmatrix} = \mathbf{i}+\mathbf{j}-2\mathbf{k} \text{ (shown)}$$

   (v)

   $$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}-1\\5\\2\end{pmatrix} \text{ from (i)}$$

   Plane OAB : $\mathbf{r}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix} = 0$ (as origin is on the plane)

   Required equation is $x-5y-2z=0$

   $$\overrightarrow{CM} = \begin{pmatrix}14\\-1\\-5\end{pmatrix} \text{ from (iv)}$$

Length of the projection of vector $\overrightarrow{CM}$ onto this plane

$$= \left| \overrightarrow{CM} \times \frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{30}} \left| \begin{pmatrix} 14 \\ -1 \\ -5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \right| = \frac{23}{\sqrt{30}} \left| \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right|$$

$$= 23\sqrt{\frac{11}{30}}$$

(vi)

Let θ be the acute angle between line OC and the triangle OAB.

$$\sin\theta = \frac{\left| \begin{pmatrix} -13 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \right|}{\sqrt{182}\sqrt{30}} = \frac{13+10+6}{\sqrt{5460}}$$

Therefore $\theta = 23.1°$

3.

$$|3\mathbf{a} - 2\mathbf{b}|^2$$
$$(3\mathbf{a} - 2\mathbf{b}) \cdot (3\mathbf{a} - 2\mathbf{b})$$
$$= 9\mathbf{a}\cdot\mathbf{a} - 6\mathbf{b}\cdot\mathbf{a} - 6\mathbf{a}\cdot\mathbf{b} + 4\mathbf{b}\cdot\mathbf{b}$$
$$= 9|\mathbf{a}|^2 - 12\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2$$
$$= 9(2)^2 - 12\mathbf{a}\cdot\mathbf{b} + 4(1)^2$$
$$= 36 - 12\mathbf{a}\cdot\mathbf{b} + 4$$
$$= 40 - 12\mathbf{a}\cdot\mathbf{b}$$

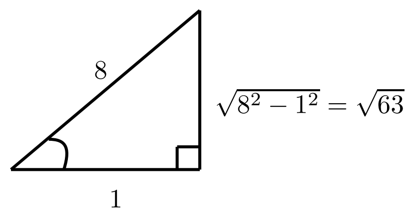
$$\therefore (\sqrt{37})^2 = 40 - 12\mathbf{a}\cdot\mathbf{b}$$
$$12\mathbf{a}\cdot\mathbf{b} = 3$$
$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{4} \text{ (shown)}$$

Since $\mathbf{b}$ is a unit vector, $|\mathbf{a}\cdot\mathbf{b}|$ is the length of projection of $\mathbf{a}$ onto $\mathbf{b}$.

(b)

It is the perpendicular distance from point A to the line $\overrightarrow{OB}$.

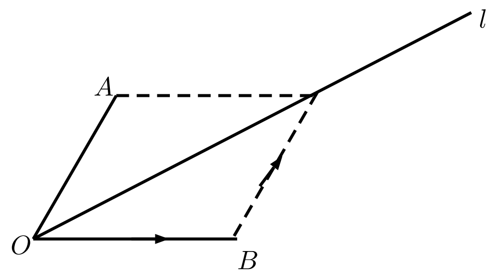
$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{4}$$
$$|\mathbf{a}||\mathbf{b}|\cos\theta = \frac{1}{4}$$
$$2(1)\cos\theta = \frac{1}{4}$$
$$\cos\theta = \frac{1}{8}$$



$$\therefore \sin\theta = \frac{\sqrt{63}}{8}$$

$$|\mathbf{a}\times\mathbf{b}| = ||\mathbf{a}||\mathbf{b}|\sin\theta|$$
$$= \left|2 \times \frac{\sqrt{63}}{8}\right|$$
$$= \frac{\sqrt{63}}{4}$$

(c)



Direction vector $l : \hat{\mathbf{a}} + \hat{\mathbf{b}}$

$\therefore l : \mathbf{r} = \lambda(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}), \lambda \in \mathbb{R}$