1. (i)

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos 60^\circ\\ &= 2(1)\left(\frac{1}{2}\right)\\ &= 1\end{aligned}$$

$$\begin{aligned}|2\mathbf{a}-\mathbf{b}|^2 &= (2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})\\ &= 4\mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}+\ \mathbf{b}\cdot\mathbf{b}\\ &= 4|\mathbf{a}|^2-4\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2\\ &= 4(2)^2-4(1)+1^2\\ &= 13\\ \therefore |2\mathbf{a}-\mathbf{b}| &= \sqrt{13}\end{aligned}$$

Similarly,

$$\begin{aligned}|2\mathbf{a}+\mathbf{b}|^2 &= 4|\mathbf{a}|^2+4\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2\\ &= 4(2)^2+4(1)+1^2\\ &= 21\\ \therefore |2\mathbf{a}+\mathbf{b}| &= \sqrt{21}\end{aligned}$$

(ii)

$$\begin{aligned}(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b}) &= |2\mathbf{a}-\mathbf{b}||2\mathbf{a}+\mathbf{b}|\cos\theta\\ 4|\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}-|\mathbf{b}|^2 &= \sqrt{13}\sqrt{21}\cos\theta\\ 4|\mathbf{a}|^2-|\mathbf{b}|^2 &= \sqrt{13}\sqrt{21}\cos\theta\\ 4(2)^2-1^2 &= \sqrt{13}\sqrt{21}\cos\theta\\ \cos\theta &= \frac{15}{\sqrt{13}\sqrt{21}}\\ \therefore \theta &\approx 24.8^\circ\end{aligned}$$

2. (a)

$$\begin{aligned}\overrightarrow{OC}\cdot\overrightarrow{OD} &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{b}\\ &= |\mathbf{a}|^2+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{a}-|\mathbf{b}|^2\\ &= |\mathbf{a}|^2-|\mathbf{b}|^2\\ &= |\mathbf{a}|^2-|\mathbf{a}|^2 \text{ (since } OA=OB)\\ &= 0\end{aligned}$$

(b)

$\overrightarrow{BA} = \overrightarrow{OA}-\overrightarrow{OB} = \mathbf{a}-\mathbf{b}$

$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}\mathbf{a}-\frac{1}{2}\mathbf{b}$

$$\begin{aligned}\text{Area } OAE &= \frac{1}{2}|\overrightarrow{AO}\times\overrightarrow{AE}|\\ &= \frac{1}{2}\left|-\mathbf{a}\times\left(\frac{1}{2}\mathbf{a}-\frac{1}{2}\mathbf{b}\right)\right|\\ &= \frac{1}{2}\left|-\frac{1}{2}\mathbf{a}\times\mathbf{a}+\frac{1}{2}\mathbf{a}\times\mathbf{b}\right|\\ &= \frac{1}{2}\left|0+\frac{1}{2}\mathbf{a}\times\mathbf{b}\right|\\ &= \frac{1}{4}|\mathbf{a}\times\mathbf{b}|\end{aligned}$$

(c)

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{OA}+\overrightarrow{AE}\\ &= \mathbf{a}+\left(\frac{1}{2}\mathbf{a}-\frac{1}{2}\mathbf{b}\right)\\ &= \frac{3}{2}\mathbf{a}-\frac{1}{2}\mathbf{b}\end{aligned}$$

$$\begin{aligned}|\overrightarrow{OE}\cdot\hat{\overrightarrow{OB}}| &= \left|\left(\frac{3}{2}\mathbf{a}-\frac{1}{2}\mathbf{b}\right)\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right|\\ &= \frac{1}{|\mathbf{b}|}\left|\frac{3}{2}\mathbf{a}\cdot\mathbf{b}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}\right|\\ &= \frac{1}{|\mathbf{b}|}\left|\frac{3}{2}\mathbf{a}\cdot\mathbf{b}-\frac{1}{2}|\mathbf{b}|^2\right|\\ &= \frac{1}{3}\left|\frac{3}{2}\mathbf{a}\cdot\mathbf{b}-\frac{3^2}{2}\right|\\ &= \frac{1}{3}\left|\frac{3}{2}\mathbf{a}\cdot\mathbf{b}-\frac{9}{2}\right|\end{aligned}$$

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\ &= 3\times 3\cos 60^\circ\\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\therefore |\overrightarrow{OE}\cdot\overrightarrow{OB}| &= \frac{1}{3}\left|\frac{3}{2}\left(\frac{9}{2}\right)-\frac{9}{2}\right|\\ &= \frac{1}{3}\left(\frac{9}{4}\right)\\ &= \frac{3}{4}\end{aligned}$$

3. (i) $\mathbf{a}\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$

$\Rightarrow \mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{a} = \mathbf{0}$

$\Rightarrow \mathbf{a}\times\mathbf{b} = \mathbf{0}$

This implies $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are parallel and since they have a common point O, they are collinear.

(ii) $\overrightarrow{OQ} = \frac{2\overrightarrow{OB}+3\overrightarrow{OA}}{5} = \frac{1}{5}(2\mathbf{b}+3\mathbf{a})$

$$
\begin{aligned}
|\overrightarrow{OQ}\cdot\hat{\mathbf{b}}| &= \left|\frac{1}{5}(2\mathbf{b}+3\mathbf{a})\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right| \\
&= \frac{1}{5|\mathbf{b}|}|2\mathbf{b}\cdot\mathbf{b}+3\mathbf{a}\cdot\mathbf{b}| \\
&= \frac{1}{5|\mathbf{b}|}|2|\mathbf{b}|^2+0| \text{ (since } \mathbf{a},\mathbf{b} \text{ are perpendicular)} \\
&= \frac{2|\mathbf{b}|^2}{5|\mathbf{b}|} \\
&= \frac{2|\mathbf{b}|}{5}
\end{aligned}
$$

Since $|\overrightarrow{OQ}\cdot\hat{\mathbf{b}}| = 3$

$\implies \frac{2}{5}|\mathbf{b}| = 3$

$\implies |\mathbf{b}| = \frac{15}{2}$

$$
\begin{aligned}
2\sqrt{14} &= \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}| \\
&= \frac{1}{2}\left|\begin{pmatrix}1\\3\\0\end{pmatrix}\times\begin{pmatrix}-1\\p\\4\end{pmatrix}\right| \\
&= \frac{1}{2}\left|\begin{pmatrix}12\\-4\\p+3\end{pmatrix}\right| \\
&= \frac{1}{2}\sqrt{12^2+4^2+(p+3)^2} \\
&= \frac{1}{2}\sqrt{160+(p+3)^2} \\
\implies 4(14) &= \frac{1}{4}(160+(p+3)^2) \\
224 &= 160+(p+3)^2 \\
(p+3)^2 &= 64 \\
p+3 &= 8 \text{ or } -8 \\
p &= 5 \text{ or } -11
\end{aligned}
$$

4. (a) Since $\mathbf{a}$ is perpendicular to $\mathbf{a}+3\mathbf{b}$,

$$
\begin{aligned}
\mathbf{a}\cdot(\mathbf{a}+3\mathbf{b}) &= 0 \\
\mathbf{a}\cdot\mathbf{a}+3\mathbf{a}\cdot\mathbf{b} &= 0 \\
|\mathbf{a}|^2+3\mathbf{a}\cdot\mathbf{b} &= 0 \\
1+3\mathbf{a}\cdot\mathbf{b} &= 0 \\
\mathbf{a}\cdot\mathbf{b} &= -\frac{1}{3}
\end{aligned}
$$

Since the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$,

$$
\begin{aligned}
\cos\frac{2\pi}{3} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\
-0.5 &= \frac{-\frac{1}{3}}{1\times|\mathbf{b}|} \\
-0.5|\mathbf{b}| &= -\frac{1}{3} \\
|\mathbf{b}| &= \frac{2}{3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
|\mathbf{b}-2\mathbf{a}|^2 &= (\mathbf{b}-2\mathbf{a})\cdot(\mathbf{b}-2\mathbf{a}) \\
&= |\mathbf{b}|^2-4\mathbf{a}\cdot\mathbf{b}+4|\mathbf{a}|^2 \\
&= \left(\frac{2}{3}\right)^2-4\left(-\frac{1}{3}\right)+4(1) \\
&= \frac{4}{9}+\frac{4}{3}+4 \\
&= \frac{52}{9} \\
|\mathbf{b}-2\mathbf{a}| &= \sqrt{\frac{52}{9}} = \frac{2\sqrt{13}}{3}
\end{aligned}
$$

(c) By ratio theorem,

$$\overrightarrow{OP} = \lambda\mathbf{b}+(1-\lambda)\mathbf{a}.$$

Area of triangle OAP

$$
\begin{aligned}
&= \frac{1}{2}|\mathbf{a}\times\mathbf{p}| \\
&= \frac{1}{2}\left|\mathbf{a}\times(\lambda\mathbf{b}+(1-\lambda)\mathbf{a})\right| \\
&= \frac{1}{2}\left|\lambda\mathbf{a}\times\mathbf{b}+(1-\lambda)\mathbf{a}\times\mathbf{a}\right| \\
&= \frac{1}{2}\lambda|\mathbf{a}\times\mathbf{b}| \\
&= \frac{\lambda}{2}\left||\mathbf{a}||\mathbf{b}|\sin\theta\right| \\
&= \frac{\lambda}{2}\left|1\left(\frac{2}{3}\right)\sin\frac{2\pi}{3}\right| \\
&= \frac{\lambda\sqrt{3}}{6}
\end{aligned}
$$

(d) $|\mathbf{a}\cdot\mathbf{b}|$ is the length of projection of $\mathbf{b}$ onto $\mathbf{a}$. $|\mathbf{a}\times\mathbf{b}|$ is the shortest distance from B to the line $\overrightarrow{OA}$.

5. (a)

$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$$

$$\overrightarrow{BC} = 2\mathbf{b}-k\mathbf{a}$$

Since B lies on AC, A, B and C are collinear, that is,

$$\lambda\overrightarrow{AB} = \overrightarrow{BC}$$
$$\lambda(\mathbf{b} - \mathbf{a}) = 2\mathbf{b} - k\mathbf{a}$$

Comparing coefficients, $\lambda = 2$.
$\therefore k = 2$.

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OB} + \overrightarrow{BC} \\ &= \mathbf{b} + 2\mathbf{b} - 2\mathbf{a} \\ &= 3\mathbf{b} - 2\mathbf{a}\end{aligned}$$

(b)

$$\begin{aligned}\overrightarrow{ON} &= (\overrightarrow{OB} \cdot \hat{\overrightarrow{OA}})\hat{\overrightarrow{OA}} \\ \frac{3}{4}\mathbf{a} &= \left(\mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} \\ \frac{3}{4}\mathbf{a} &= \frac{(\mathbf{b} \cdot \mathbf{a})}{|\mathbf{a}|^2}\mathbf{a} \\ \frac{3}{4}\mathbf{a} &= \frac{4}{|\mathbf{a}|^2}\mathbf{a}\end{aligned}$$

Comparing coefficient,

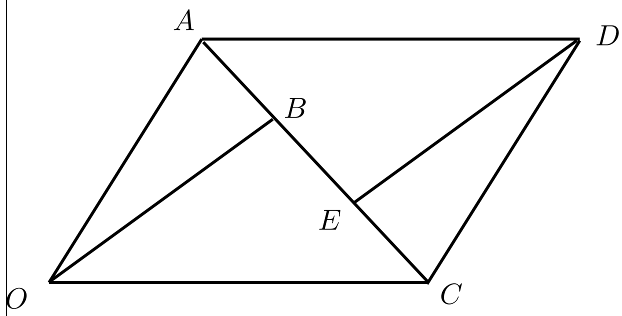
$$\begin{aligned}\frac{3}{4} &= \frac{4}{|\mathbf{a}|^2} \\ |\mathbf{a}|^2 &= \frac{16}{3} \\ |\mathbf{a}| &= \frac{4}{\sqrt{3}}\end{aligned}$$

(c)

$$\begin{aligned}\cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a} \cdot \mathbf{b}|} \\ \cos\theta &= \frac{4}{\frac{4}{\sqrt{3}} \times \sqrt{2^2 + 2^2 + 2^2}} \\ \theta &= 60^\circ\end{aligned}$$

(d) Since $\overrightarrow{BC} = 2\mathbf{b} - 2\mathbf{a}$ (from (a)) and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, we have $\overrightarrow{BC} = 2\overrightarrow{AB}$.

Hence, the diagram of the parallelogram is as follows:



From the diagram above,

$$\begin{aligned}\overrightarrow{DE} &= \overrightarrow{BO} \\ &= -\overrightarrow{OB} \\ &= -\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}\end{aligned}$$

6. (a)

$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{BQ} \\ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} &= \overrightarrow{OQ} - \overrightarrow{OB} \\ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} &= \overrightarrow{OQ} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \\ \overrightarrow{OQ} &= \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}\end{aligned}$$

(b) It is the projection vector of $\mathbf{a}$ onto $\mathbf{b}$.

(c)

$$\begin{aligned}|\mathbf{a}| &< |\mathbf{b}| \\ \sqrt{p^2 + (p-1)^2 + 2^2} &< \sqrt{1 + 2^2 + 2^2} \\ \sqrt{2p^2 - 2p + 5} &< 3 \\ 2p^2 - 2p + 5 &< 9 \\ 2p^2 - 2p - 4 &< 0 \\ p^2 - p - 2 &< 0 \\ (p-2)(p+1) &< 0 \\ -1 < p &< 2\end{aligned}$$

Since $p > 0$, we obtain $0 < p < 2$.

(d) When $p = 2$,

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}2\\1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\4\end{pmatrix}$$

$$\mathbf{a}-\mathbf{b} = \begin{pmatrix}2\\1\\2\end{pmatrix} - \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \begin{pmatrix}3\\-1\\4\end{pmatrix} \cdot \begin{pmatrix}1\\3\\0\end{pmatrix} = 3 - 3 + 0 = 0$$

$\therefore$ $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are perpendicular.

Note that **a** + **b** and **a** - **b** are the diagonals of the parallelogram $OAQB$.

The diagonals are perpendicular implies that the parallelogram is a rhombus.

Thus, $|\mathbf{a}\times\mathbf{b}|$ is the area of the rhombus $OAQB$.