Vectors J1 EOY Revision

1. [2016/NYJC/Prelim/I/11]

   The line l_1 passes through the point A with coordinates $(1, 2, 1)$ and is parallel to the vector $\mathbf{i} + a\mathbf{j} + 2\mathbf{k}$, where $a \in \mathbb{R}$. The line l_2 has equation $x - 3 = \frac{y}{2} = \frac{z-5}{3}$. It is given that l_1 and l_2 intersect at point B.

   i. Find the value of a. [4]

   ii. The plane p_1 contains the point A and is perpendicular to l_2. Find the exact shortest distance from point B to p_1. Hence find the acute angle between l_1 and p_1. [5]

   iii. Find a cartesian equation of plane p_2 that is perpendicular to p_1 and contains l_1. [3]

   iv. Find the acute angle between p_2 and the xy-plane. [2]

   [(i) $a = -1$ (ii) $\frac{10}{\sqrt{14}}$; $\theta = 33.1°$ (iii) $7x + y - 3z = 6$ (iv) $\alpha = 67.0°$]

2. [2016/SAJC/Prelim/I/7]

   Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha > 1$ and $\beta > 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{5\pi}{6}$.

   i. Show that $|\mathbf{b}| = \frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$. [3]

   ii. Give the geometrical interpretation of $|\mathbf{a} \cdot \mathbf{b}|$ and find its value in terms of α and β. [3]

   iii. The point M divides AB in the ratio $\lambda : 1 - \lambda$ where $0 < \lambda < 1$. The point N is such that $OMNB$ is a parallelogram. Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and the area of the triangle OAN in terms of λ, α and β. [5]

   [(ii) $|\mathbf{a} \cdot \mathbf{b}| = \left(\frac{\alpha}{\beta}\right)$ (iii) $\overrightarrow{ON} = (\lambda + 1)\mathbf{b} + (1 - \lambda)\mathbf{a}$; Area of triangle OAN= $\frac{(\lambda+1)\sqrt{3}}{6}\left(\frac{\alpha}{\beta}\right)$]

3. [2016/TJC/Prelim/I/4]

   Relative to the origin O, the points A, B, M and N have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{m}$ and $\mathbf{n}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. It is given that $\mathbf{m} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ and $\mathbf{n} = 2(1 - \lambda)\mathbf{a} - \lambda\mathbf{b}$ where λ is a real parameter.

   i. Show that $\mathbf{m} \times \mathbf{n} = (3\lambda^2 - 4\lambda + 2)(\mathbf{b} \times \mathbf{a})$. [2]

   ii. It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$. Find the smallest area of the triangle MON as λ varies. [4]

   [(ii) 2 units^2]

4. [2016/TJC/Prelim/II/3]

   The line l has equation $\frac{x+1}{-1} = \frac{z+6}{2}, y = 4$ and the point A has coordinates $(-1, 3, -5)$.

   i. Find the position vector of the foot of the perpendicular from A to l. [3]

   ii. Plane p_1 contains l and A. Show that the equation of p_1 is $2x + y + z = -4$. [3]

   iii. Given that the plane p_2 has equation $x + 2y + cz = -5$ where c is a negative constant, and that the acute angle between p_1 and p_2 is $60°$, find the value of c. [3]

   iv. Find the equation of the line of intersection, m, between p_1 and p_2. [1]

   [(i) $\frac{1}{5}\begin{pmatrix}-7\\20\\-26\end{pmatrix}$ (iii) $c = -1$ (iv) $\mathbf{r} = \begin{pmatrix}-1\\-2\\0\end{pmatrix} + t\begin{pmatrix}-1\\1\\1\end{pmatrix}$]

5. [2014/NYJC/Prelim/I/9]

   i. The lines l_1 and l_2 meet at the point P. The line l_3 is coplanar with l_1 and l_2 and is perpendicular to l_1. Given that l_1 and l_2 are parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, show that l_3 is parallel to the vector $\mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$. [3]

   ii. The equations of l_1 and l_2 are now known to be $\mathbf{r} = \begin{pmatrix}3\\-5\\2\end{pmatrix} + t\begin{pmatrix}11\\10\\2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}3\\-5\\2\end{pmatrix} + s\begin{pmatrix}17\\3\\4\end{pmatrix}$ respectively, where s and t are real parameters. Find the equation of the line l_3, given that l_3 also passes through P. [2]

   iii. The line l_4 has equation $\mathbf{r} = \begin{pmatrix}10\\-3\\1\end{pmatrix} + u\begin{pmatrix}-3\\4\\-1\end{pmatrix}$, where u is a real parameter. Determine if l_3 and l_4 are skew or intersecting. [3]

   iv. The line l_5 is perpendicular to both l_3 and l_4. Find the acute angle between l_5 and the plane containing l_1 and l_2. [5]

   [(ii) $\mathbf{r} = \begin{pmatrix}3\\-5\\2\end{pmatrix} + \lambda\begin{pmatrix}6\\-7\\2\end{pmatrix}$, where $\lambda \in \mathbb{R}$ (iii) Skew (iv) 83.9°]

6. [2014/AJC/Prelim/I/8]

   The equations of planes P_1, P_2 are

$$P_1 : \mathbf{r} = \begin{pmatrix}-3\\10\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\12\\3\end{pmatrix} + \beta\begin{pmatrix}1\\1\\1\end{pmatrix}, \lambda, \beta \in \mathbb{R} \qquad P_2 : \mathbf{r}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = 1$$

   i. Find the coordinates of the foot of perpendicular from point $A(-3, 10, 3)$ to the plane P_2 and show that the point $B(2, 10, -2)$ is the reflection of point A in P_2. [5]

   ii. The planes P_1 and P_2 meet in a line L. Find a vector equation of line L. [3]

   iii. Plane P_3 is the reflection of P_1 in P_2. Using the results above, find a vector perpendicular to P_3. Hence find, in scalar product form, the equation of P_3. [3]

   [(i) $(-\frac{1}{2}, 10, \frac{1}{2})$; (ii) $\mathbf{r} = \begin{pmatrix}-1\\-2\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\1\end{pmatrix}, t \in \mathbb{R}$ (iii) $\mathbf{r}\cdot\begin{pmatrix}14\\-5\\-9\end{pmatrix} = -4$]

7. [2014/TJC/Prelim/I/12]

   Two planes p_1 and p_2 have equations $ax - 3y - z = b$ and $4x + y + bz = 2a$ respectively. They intersect at the line l which contains the point $A(1, 0, -1)$.

   i. Find the values of a and b. [2]

   ii. Without the use of a graphic calculator, find a vector equation of the line l. [2]

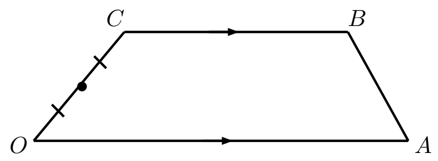
   iii. Given that the point $N(-4, -6, 12)$ is the foot of perpendicular from point $B(1, c, d)$ to the line l, show that $6c - 13d = -217$. [3]

   iv. Another plane p_3 is parallel to the plane p_2 and contains B. Given that the distance between planes p_3 and p_2 is $\frac{5}{\sqrt{21}}$. Find the values of c and d. [5]

   v. Hence write down two possible equations of plane p_3. [2]

[(i) $a = 1, b = 2$ (ii) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix}$ (iv) $c = -21, d = 7$; $c = -15.8, d = 9.4$ (v) $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = -3$; $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 7$]

8. [2014/VJC/Prelim/I/9a]



$OABC$ is a trapezium such that CB is parallel to OA and $CB : OA = k : 1$, where k is a constant and $0 < k < 1$.

It is given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, and M is the midpoint of OC. Find $\overrightarrow{OB}$ in terms of k, $\mathbf{a}$ and $\mathbf{c}$ and show that the area of triangle AMB can be written as $\lambda|\mathbf{a} \times \mathbf{c}|$, where λ is a constant to be found in terms of k. [4]

[$\lambda = \frac{1}{4}(k+1)$]

9. [2014/DHS/Prelim/II/3]

Referred to the origin O, $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors denoting the position vectors of the points A and B repsectively.

If a point C has a position vector $\mathbf{c}$ such that $\mathbf{c} = (\mathbf{a} \cdot \mathbf{a})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$,

i. what can you say about the points O, A, B, and C? [1]

ii. Given that $OBCA$ is a parallelogram, find $|\mathbf{a}|$. [2]

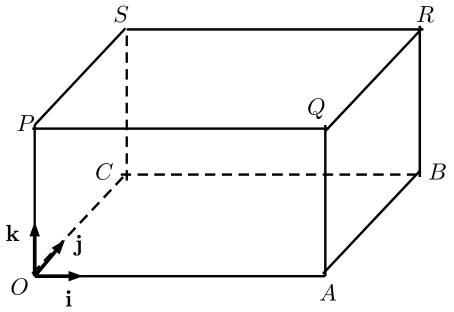
The point U is the mid-point of OA and the point V on OB is such that $OV : VB = 3 : 2$. The point N lies on UV such that $UN : UV = 2 : 3$.

iv. Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

v. If the angle AOB is $\frac{\pi}{3}$, find the area of triangle OUN, giving your answer in terms of $|\mathbf{b}|$. [4]

[(ii) 1 (iii) $\frac{1}{6}\mathbf{a} + \frac{2}{5}\mathbf{b}$ (iv) $\frac{\sqrt{3}}{20}|\mathbf{b}|$]

10. [2014/RI/Prelim/I/9]



The diagram shows a cuboid with horizontal rectangular base $OABC$, where $OA = 5$ units, $OC = 3$ units and $OP = 2$ units. The edges OP, AQ, BR and CS are vertical, and $PQRS$ is the top of the cuboid. Using O as the origin, unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are taken along OA, OC and OP respectively.

The plane π_1 contains the points A, C and R, and the plane π_2 contains the points Q, S and B.

i. Find, in scalar product form, an equation of π_1. [3]

ii. Find the acute angle between π_1 and the horizontal base. [2]

iii. Hence state the acute angle between π_1 and π_2. [1]

The point X lies on PS such that $\overrightarrow{SX} = \alpha\overrightarrow{SP}$, where $\alpha > 0$ is a constant, and Y is the point with coordinates $(5, 2, 1)$.

iv. Find the equations of the lines XY and OR in vector form. [3]

v. Given that the lines XY and OR intersect at a point W, find α and the ratio $OW : OR$. [4]

[(i) $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 10 \\ -15 \end{pmatrix} = 30$ (ii) $37.9°$ (iii) $75.7°$ (iv) $l_{XY}: \quad \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3\alpha - 1 \\ -1 \end{pmatrix}$; $l_{OR}: \quad \mathbf{r} = \mu \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$
(v) $OW : OR = 2 : 3$]