1. (i)

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda \begin{pmatrix}1\\a\\2\end{pmatrix}, \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \begin{pmatrix}3\\0\\5\end{pmatrix} + \mu \begin{pmatrix}1\\2\\3\end{pmatrix}, \mu \in \mathbb{R}$$

Equating the equations of the 2 lines,

$1 + \lambda = 3 + \mu \ldots (1)$

$2 + a\lambda = 2\mu \ldots (2)$

$1 + 2\lambda = 5 + 3\mu \ldots (3)$

Solving, we get $\mu = 0, \lambda = 2$.

Substituting into equation 2, we get $a = -1$.

(ii)

Since p_1 is perependicular to l_2, this implies that the normal vector of p_1 is $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Coordinates of B is $(3, 0, 5)$

$$\begin{aligned}\text{Shortest distance from } B \text{ to } p_1 &= \left|\overrightarrow{AB} \cdot \hat{\mathbf{n_1}}\right| \\ &= \frac{\left|\overrightarrow{AB} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} \\ &= \frac{\left|\left[\begin{pmatrix}3\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} \\ &= \frac{\left|\begin{pmatrix}2\\-2\\4\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} \\ &= \frac{10}{\sqrt{14}}\end{aligned}$$

$|\overrightarrow{AB}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$

$\sin\theta = \frac{10}{\sqrt{14}} \div \sqrt{24}$

$\theta \approx 33.1°$

(iii)

A normal vector to p_2 is $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}7\\1\\-3\end{pmatrix}$

Equation of p_2 is $\mathbf{r} \cdot \begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}7\\1\\-3\end{pmatrix} = 6$

Cartesian equation of p_2 is $7x + y - 3z = 6$

(iv)

Equation of $x - y$ plane is $z = 0$. Hence its normal vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let α be the angle between p_2 and $x - y$ plane.

$$\begin{aligned} \cos\alpha &= \frac{\left| \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{59}} \\ &= \frac{3}{\sqrt{59}} \\ \alpha &= 67.0^\circ \end{aligned}$$

2. (i)

$$\begin{aligned} \mathbf{a} \cdot (\alpha\mathbf{a} + \beta\mathbf{b}) &= 0 \\ \alpha|\mathbf{a}|^2 + \beta\mathbf{a} \cdot \mathbf{b} &= 0 \\ \alpha + \beta\mathbf{a} \cdot \mathbf{b} &= 0 \\ \mathbf{a} \cdot \mathbf{b} &= -\frac{\alpha}{\beta} \end{aligned}$$

Since angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{5\pi}{6}$,

$$\begin{aligned} \cos\left(\frac{5\pi}{6}\right) &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ -\frac{\sqrt{3}}{2} &= \frac{-\frac{\alpha}{\beta}}{|\mathbf{b}|} \\ |\mathbf{b}| &= \frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right) \text{ (shown)} \end{aligned}$$

(ii)

$|\mathbf{a} \cdot \mathbf{b}|$ is the length of projection of $\mathbf{b}$ onto $\mathbf{a}$

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= \left| |\mathbf{a}||\mathbf{b}| \cos\left(\frac{5\pi}{6}\right) \right| \\ &= |\mathbf{b}|\frac{\sqrt{3}}{2} \\ &= \frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)\frac{\sqrt{3}}{2} \\ &= \left(\frac{\alpha}{\beta}\right) \end{aligned}$$

(ii)

By ratio theorem,

$\overrightarrow{OM} = \lambda\mathbf{b} + (1 - \lambda)\mathbf{a}$

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OM} + \overrightarrow{MN}\\ &= [\lambda\mathbf{b} + (1-\lambda)\mathbf{a}] + \overrightarrow{OB}\\ &= [\lambda\mathbf{b} + (1-\lambda)\mathbf{a}] + \mathbf{b}\\ &= (\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}\end{aligned}$$

Area of triangle OAN

$$\begin{aligned}&= \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{ON}|\\ &= \frac{1}{2}|\mathbf{a} \times [(\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}]|\\ &= \frac{1}{2}|(\lambda+1)\mathbf{a}\times\mathbf{b} + (1-\lambda)\mathbf{a}\times\mathbf{a}|\\ &= \frac{1}{2}|(\lambda+1)\mathbf{a}\times\mathbf{b}|\\ &= \frac{1}{2}(\lambda+1)|\mathbf{a}\times\mathbf{b}| \quad \text{since } |\lambda+1| = \lambda+1 \text{ as } 0<\lambda<1\\ &= \frac{1}{2}(\lambda+1)|\mathbf{a}||\mathbf{b}|\left|\sin\left(\frac{5\pi}{6}\right)\right|\\ &= \frac{(\lambda+1)}{2}\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{\alpha}{\beta}\right)\left(\frac{1}{2}\right)\\ &= \frac{(\lambda+1)\sqrt{3}}{6}\left(\frac{\alpha}{\beta}\right)\end{aligned}$$

3. (i)

$$\begin{aligned}\mathbf{m}\times\mathbf{n} &= (\lambda\mathbf{a} + (1-\lambda)\mathbf{b} \times (2(1-\lambda)\mathbf{a} - \lambda\mathbf{b})\\ &= 2\lambda(1-\lambda)(\mathbf{a}\times\mathbf{a}) - \lambda^2(\mathbf{a}\times\mathbf{b}) + 2(1-\lambda)^2(\mathbf{b}\times\mathbf{a}) - \lambda(1-\lambda)(\mathbf{b}\times\mathbf{b})\\ &= 2(1-\lambda)^2(\mathbf{b}\times\mathbf{a}) - \lambda^2(\mathbf{a}\times\mathbf{b}) \quad \text{since } \mathbf{a}\times\mathbf{a} = 0 = \mathbf{b}\times\mathbf{b}\\ &= (2(1-\lambda)^2 + \lambda^2)(\mathbf{b}\times\mathbf{a}) \quad \text{since } \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}\\ &= (3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})\end{aligned}$$

(ii)

$$\begin{aligned}\text{Area of triangle } MON = \frac{1}{2}|\mathbf{m}\times\mathbf{n}| &= \frac{1}{2}|(3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})|\\ &= \frac{1}{2}|(3\lambda^2 - 4\lambda + 2||\mathbf{b}\times\mathbf{a}|\\ &= \frac{1}{2}\left|3\left(\lambda - \frac{2}{3}\lambda\right)^2 + \frac{2}{3}\right||\mathbf{b}||\mathbf{a}|\sin\frac{\pi}{6} \quad \text{(by completing the square)}\\ &= 3\left|3\left(\lambda - \frac{2}{3}\right)^2 + \frac{2}{3}\right|\end{aligned}$$

$\therefore$ smallest area is $3 \times \frac{2}{3} = 2$ units^2

4. (i)

Let $\lambda = \frac{x+1}{-1} = \frac{z+6}{2}\, y = 4$

$x = -1-\lambda,\ z = -6+2\lambda,\ y = 4$

$l : \mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

Let N be the foot of the perpendicular from A to l.

$\overrightarrow{ON} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ 4 \\ -6+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$\Rightarrow \overrightarrow{AN} = \begin{pmatrix} -\lambda \\ 1 \\ -1+2\lambda \end{pmatrix}$

$\overrightarrow{AN} \perp l \quad \Longrightarrow \begin{pmatrix} -\lambda \\ 1 \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 0 \Rightarrow \lambda = \frac{2}{5}$

Thus $\overrightarrow{ON} = \frac{1}{5} \begin{pmatrix} -7 \\ 20 \\ -26 \end{pmatrix}$

(ii)

Let B be the point on l with coordinates $(-1, 4, -6)$

$\overrightarrow{BA} = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

A normal to p_1 is $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Equation of p_1 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = -4$

$2x + y + z = -4$ (shown)

(iii)

Equation of p_2 is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix} = -5$

$$\begin{aligned}
\cos 60^\circ &= \frac{\left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix} \right|}{\sqrt{6}\sqrt{5+c^2}} \\
\frac{1}{2} &= \frac{|4+c|}{\sqrt{6}\sqrt{5+c^2}} \\
30 + 6c^2 &= 4(c^2 + 8c + 16)
\end{aligned}$$

$c^2 - 16c - 17 = 0$

$(c-17)(c+1) = 0$

$c = 17$ (rejected since $c < 0$) or $c = -1$

(iv)

$p_1 : 2x + y + z = -4$

$p_2 : x + 2y + cz = -5$

Using GC,

$x = -1 - \mathbf{t}$

$y = -2 + \mathbf{t}$

$z = \mathbf{t}$

Equation of m is $\mathbf{r} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, t \in \mathbb{R}$

5. i. Since l_3 is coplanar to l_1 and l_2, l_3 is parallel to the vector $s\mathbf{b} + t\mathbf{a}$ for a fixed value of $s, t \in \mathbb{R}$.

Also, l_3 is perpendicular to l_1, so

$$\begin{aligned} \mathbf{a} \cdot (s\mathbf{b} + t\mathbf{a}) &= 0 \\ s\mathbf{a} \cdot \mathbf{b} + t\mathbf{a} \cdot \mathbf{a} &= 0 \\ t|\mathbf{a}|^2 &= -s\mathbf{a} \cdot \mathbf{b} \\ t &= \frac{-s\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \end{aligned}$$

Substituting t into $s\mathbf{b} + t\mathbf{a}$, then l_3 is parallel to

$$s\mathbf{b} + \left(\frac{-s\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = s \left[\mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} \right] \quad \text{(shown)}$$

ii. From part (i), direction vector of l_3 is given by

$$\mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \begin{pmatrix} 17 \\ 3 \\ 4 \end{pmatrix} - \frac{\begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ 3 \\ 4 \end{pmatrix}}{\left| \begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix} \right|^2} \begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix}$$

Equation of l_3: $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

iii. Since $\begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$ for any $k \in \mathbb{R}$, lines l_3 and l_4 are not parallel.

Equating both lines together, we obtain

$$\begin{pmatrix} 3 + 6\lambda \\ -5 - 7\lambda \\ 2 + 2\lambda \end{pmatrix} = \begin{pmatrix} 10 - 3\mu \\ -3 + 4\mu \\ 1 - \mu \end{pmatrix},$$

Solving simultaneous equations using GC, there is no solution for λ and μ. So, l_3 and l_4 are non-parallel and non-intersecting, hence skew lines.

iv. Direction vector of l_5:

$$\mathbf{m_3} \times \mathbf{m_4} = \begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

Normal vector of plane:

$$\mathbf{m_1} \times \mathbf{m_2} = \begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix} \times \begin{pmatrix} 17 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -34 \\ 10 \\ 137 \end{pmatrix}.$$

Required angle θ:

$$\sin\theta = \frac{\left| \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -34 \\ 10 \\ 137 \end{pmatrix} \right|}{\sqrt{1+3^2}\sqrt{34^2+10^2+137^2}}$$

$$\theta = 83.9^\circ$$

6. i. We first form the line that passes through A and perpendicular to plane P_2.

$$\mathbf{r} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

Intersecting this line with P_2:

$$\begin{pmatrix} -3-\lambda \\ 10 \\ 3+\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$(3+\lambda) + (3+\lambda) = 1$$

$$\lambda = -\frac{5}{2}$$

$$\therefore \text{ foot of perpendicular } \overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \tfrac{5}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix}$$

Coordinate: $(-\frac{1}{2}, 10, \frac{1}{2})$.

By ratio theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

$$\overrightarrow{OB} = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$$

Hence, $(2, 10, -2)$ is the reflection of $(-3, 10, 3)$ in P_3.

ii. We first find the scalar-product form for P_1.

$$\mathbf{n} = \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix}$$

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

$$\mathbf{r} \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = -19$$

$$\begin{aligned} 9x + 5y - 14z &= -19 \\ -x + z &= 1 \end{aligned}$$

Using GC to solve P_1 and P_2: $\mathbf{r} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, t \in \mathbb{R}$.

iii. To find P_3, we first need to obtain 2 vectors parallel to P_3, so that we can form its normal vector.

Since L lies on P_3, P_3 is parallel to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Note that P_3 contains the point $\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$ (from L) and point B $\begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$.

Hence P_3 is parallel to $\begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix}$

Hence, normal vector of P_3:

$$\mathbf{n_3} = \begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}$$

Equation of P_3:

$$\begin{aligned} \mathbf{r} \cdot \mathbf{n} &= \mathbf{a} \cdot \mathbf{n} \\ \mathbf{r} \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} &= \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} \\ \mathbf{r} \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} &= -4 \end{aligned}$$

7. i. Since $A(1, 0, -1)$ lies on both planes, we subst this point into the planes.

$a(1) - 3(0) - (-1) = b \implies a - b = 1 \ldots (1)$

$4(1) + 0 + b(-1) = 2a \implies 2a + b = 4 \ldots (2)$

Using GC to solve for (1) and (2), $a = 1, b = 2$.

ii. Direction vector of line $l = \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 13 \end{pmatrix}$

Vector equation of line $l : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

iii. Since N is the foot of perpendicular from point B to l, $\overrightarrow{BN}$ is perpendicular to l.

$$\overrightarrow{BN} \cdot \mathbf{m} = 0$$
$$(\overrightarrow{ON} - \overrightarrow{OB}) \cdot \mathbf{m} = 0$$
$$\left(\begin{pmatrix} -4 \\ -6 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}\right) \cdot \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = 0$$
$$\begin{pmatrix} -5 \\ -6-c \\ 12-d \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = 0$$
$$-25 - 36 - 6c - 156 + 13d = 0$$
$$6c - 13d = -217 \text{ (shown)}$$

iv. P_2 contains l, and l contains N, hence P_2 contains N.

Also, B is on P_3.

Hence, the required distance is the projection of $\overrightarrow{BN}$ onto the normal vector.

$$\frac{\left|\overrightarrow{BN} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}\right|}{\sqrt{4^2 + 1 + 2^2}} = \frac{5}{\sqrt{21}}$$
$$\left|\left[\begin{pmatrix} -4 \\ -6 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}\right] \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}\right| = 5$$
$$|-20 - 6 - c + 24 - 2d| = 5$$
$$|-2 - c - 2d| = 5$$

$-2 - c - 2d = 5$

$-c - 2d = 7 \ldots (1)$
$6c - 13d = -217 \ldots (2)$

Solving (1) and (2), we get $c = -21, d = 7$

$2 + c + 2d = 5$

$c + 2d = 3 \ldots (3)$
$6c - 13d = -217 \ldots (4)$

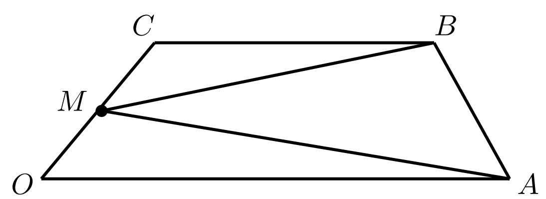
Solving (3) and (4), we get $c = -15.8, d = 9.4$

v. Using formula $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, equations of P_3 are:

$$\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -21 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 4 - 21 + 14 = -3$$

$$\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -15.8 \\ 9.4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 4 - 15.8 + 18.8 = 7$$

8.



$\overrightarrow{CB} = k\mathbf{a}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \mathbf{c} + k\mathbf{a}$

$\overrightarrow{OM} = \frac{1}{2}\mathbf{c}$

$\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM} = \mathbf{c} + k\mathbf{a} - \frac{1}{2}\mathbf{c} = \frac{1}{2}\mathbf{c} + k\mathbf{a}$

$\overrightarrow{MA} = \overrightarrow{OA} - \overrightarrow{OM} = \mathbf{a} - \frac{1}{2}\mathbf{c}$

$$\begin{aligned}
&\text{area of triangle } AMB\\
=&\frac{1}{2}\left|\overrightarrow{MB} \times \overrightarrow{MA}\right|\\
=&\frac{1}{2}\left|\left(\frac{1}{2}\mathbf{c} + k\mathbf{a}\right) \times \left(\mathbf{a} - \frac{1}{2}\mathbf{c}\right)\right|\\
=&\frac{1}{2}\left|\frac{1}{2}(\mathbf{c} \times \mathbf{a}) - \frac{1}{4}(\mathbf{c} \times \mathbf{c}) + k(\mathbf{a} \times \mathbf{a}) - \frac{k}{2}(\mathbf{a} \times \mathbf{c})\right|\\
=&\frac{1}{2}\left|\frac{1}{2}(\mathbf{c} \times \mathbf{a}) - \frac{k}{2}(\mathbf{a} \times \mathbf{c})\right| \quad (\text{since } \mathbf{a} \times \mathbf{a} = \mathbf{c} \times \mathbf{c} = 0)\\
=&\frac{1}{2}\left|\frac{1}{2}(\mathbf{c} \times \mathbf{a}) + \frac{k}{2}(\mathbf{c} \times \mathbf{a})\right|\\
=&\frac{1}{4}|k+1||\mathbf{a} \times \mathbf{c}|\\
=&\frac{1}{4}(k+1)|\mathbf{a} \times \mathbf{c}| \quad \text{where } |k+1| = (k+1) \text{ since } k > 0.
\end{aligned}$$

9. i. O, A, B and C lie on the same plane.

ii. Since $OBCA$ is a parallelogram , we have $\overrightarrow{AC} = \overrightarrow{OB}$.

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{OB}\\
\mathbf{c} - \mathbf{a} &= \mathbf{b}\\
\mathbf{c} &= \mathbf{a} + \mathbf{b}\\
(\mathbf{a} \cdot \mathbf{a})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{b} &= \mathbf{a} + \mathbf{b}
\end{aligned}$$

By comparing coefficients of $\mathbf{a}$,

$|\mathbf{a}|^2 = 1 \quad \Longrightarrow |\mathbf{a}| = 1.$

iii. By ratio theorem,

$$\begin{aligned}
\overrightarrow{ON} &= \frac{\overrightarrow{OU} + 2\overrightarrow{OV}}{3}\\
&= \frac{1}{3}\left(\frac{1}{2}\mathbf{a} + \frac{6}{5}\mathbf{b}\right)\\
&= \frac{1}{6}\mathbf{a} + \frac{2}{5}\mathbf{b}
\end{aligned}$$

iv.

$$\begin{aligned}
\text{Area } OUN &= \frac{1}{2}|\overrightarrow{OU} \times \overrightarrow{ON}|\\
&= \frac{1}{2}\left|\frac{1}{2}\mathbf{a} \times \left(\frac{1}{6}\mathbf{a} + \frac{2}{5}\mathbf{b}\right)\right|\\
&= \frac{1}{2}\left|\frac{1}{12}(\mathbf{a} \times \mathbf{a}) + \frac{1}{5}(\mathbf{a} \times \mathbf{b})\right|\\
&= \frac{1}{10}|\mathbf{a} \times \mathbf{b}|\\
&= \frac{1}{10}|\mathbf{a}||\mathbf{b}|\left|\sin\frac{\pi}{3}\right|\\
&= \frac{1}{10}\left(\frac{\sqrt{3}}{2}\right)|\mathbf{b}|\\
&= \frac{\sqrt{3}}{20}|\mathbf{b}|
\end{aligned}$$

10. i. $\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix}5\\3\\2\end{pmatrix} - \begin{pmatrix}5\\0\\0\end{pmatrix} = \begin{pmatrix}0\\3\\2\end{pmatrix}$

$\overrightarrow{CR} = \overrightarrow{OR} - \overrightarrow{OC} = \begin{pmatrix}5\\3\\2\end{pmatrix} - \begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}5\\0\\2\end{pmatrix}$

$\mathbf{n} = \overrightarrow{AR} \times \overrightarrow{CR} = \begin{pmatrix}0\\3\\2\end{pmatrix} \times \begin{pmatrix}5\\0\\2\end{pmatrix} = \begin{pmatrix}6\\10\\-15\end{pmatrix}$

Equation of π_1:

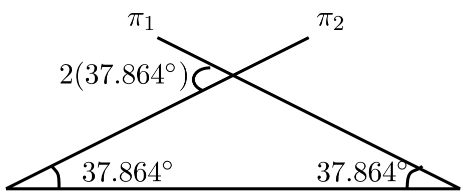
$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

$$\mathbf{r} \cdot \begin{pmatrix}6\\10\\-15\end{pmatrix} = \begin{pmatrix}5\\0\\0\end{pmatrix} \cdot \begin{pmatrix}6\\10\\-15\end{pmatrix} = 30$$

ii. Note that the normal vector of the horizontal base is $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

$$\text{Required angle} = \cos^{-1} \frac{\left|\begin{pmatrix}6\\10\\-15\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{36+100+225}\sqrt{0+0+1}} = \cos^{-1}\left(\frac{15}{19}\right) = 37.864° \approx 37.9°$$

iii. By symmetry, the acute angle between π_2 and the horizontal base is also $37.864°$.



Hence, the acute angle between π_1 and π_2 is $2(37.864°) = 75.7°$

iv. $\overrightarrow{OX} = \overrightarrow{OS} + \overrightarrow{SX} = \begin{pmatrix}0\\3\\2\end{pmatrix} + \alpha \begin{pmatrix}0\\-3\\0\end{pmatrix} = \begin{pmatrix}0\\3(1-\alpha)\\2\end{pmatrix}$

$l_{XY}: \quad \mathbf{r} = \overrightarrow{OY} + \lambda \overrightarrow{XY}, \lambda \in \mathbb{R}$

$$\mathbf{r} = \begin{pmatrix}5\\2\\1\end{pmatrix} + \lambda \begin{pmatrix}5\\3\alpha - 1\\-1\end{pmatrix}, \lambda \in \mathbb{R}.$$

$l_{OR}: \quad \mathbf{r} = \mu \begin{pmatrix}5\\3\\2\end{pmatrix}, \mu \in \mathbb{R}$

v. We equate lines XY and OR together to find W.

$$\mu \begin{pmatrix}5\\3\\2\end{pmatrix} = \begin{pmatrix}5\\2\\1\end{pmatrix} + \lambda \begin{pmatrix}5\\3\alpha - 1\\-1\end{pmatrix}$$

We obtain the following 3 equations with 3 unknowns:

$$\mu = 1 + \lambda$$
$$3\mu = 2 + \lambda(3\alpha - 1)$$
$$2\mu = 1 - \lambda$$

Using GC, we obtain $\mu = \frac{2}{3}, \lambda = -\frac{1}{3}$ and $\alpha = \frac{1}{3}$.

Since $\overrightarrow{OW} = \mu\overrightarrow{OR}$, we obtain $OW : OR = 2 : 3$.

