1.

$$\begin{aligned}
w &= x^2 y\\
\frac{dw}{dx} &= 2xy + x^2\frac{dy}{dx}\\
\frac{1}{x}\frac{dw}{dx} &= 2y + x\frac{dy}{dx}\\
x\frac{dy}{dx} &= \frac{1}{x}\frac{dw}{dx} - 2y
\end{aligned}$$

Substitute $x\frac{dy}{dx} = \frac{1}{x}\frac{dw}{dx} - 2y$ and $y = \frac{w}{x^2}$ into $x\frac{dy}{dx} + 2y + 3xy = 0$,

$$\begin{aligned}
x\frac{dy}{dx} + 2y + 3xy &= 0\\
\frac{1}{x}\frac{dw}{dx} - 2y + 2y + 3x\left(\frac{w}{x^2}\right) &= 0\\
\frac{1}{x}\frac{dw}{dx} + \left(\frac{3w}{x}\right) &= 0\\
\frac{dw}{dx} &= -3w \text{ (Shown)}
\end{aligned}$$

$$\begin{aligned}
&\frac{dw}{dx} = -3w\\
\implies &\int \frac{1}{w}\,dw = -3\int 1\,dx\\
\implies &\ln|w| = -3x + C\\
\implies &|w| = e^{-3x+C}\\
\implies &w = \pm e^{-3x}e^{C}\\
\implies &w = Ae^{-3x}\\
\implies &x^2 y = Ae^{-3x}
\end{aligned}$$

Given that $y = -\frac{1}{2}$ when $x = 2$,

$$\begin{aligned}
2^2\left(-\frac{1}{2}\right) &= Ae^{-6}\\
\implies A &= -2e^{6}
\end{aligned}$$

$$\therefore x^2 y = -2e^{6}e^{-3x} \implies y = \frac{-2e^{6-3x}}{x^2}$$

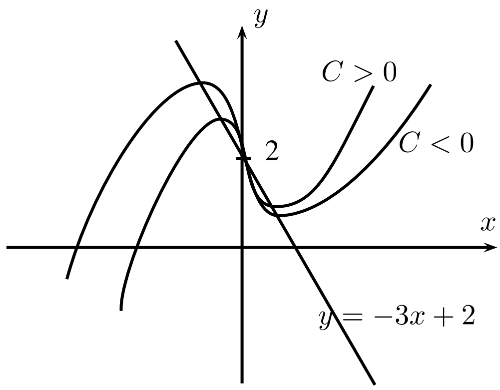
2. (a)

$$y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\begin{aligned}
&x\frac{dy}{dx} = 3x + y - 2\\
\implies &x\left(v + x\frac{dv}{dx}\right) = 3x + vx - 2\\
\implies &xv + x^2\frac{dv}{dx} = 3x + vx - 2\\
\implies &\frac{dv}{dx} = \frac{3x-2}{x^2}\\
\implies &\int dv = \int \frac{3}{x} - \frac{2}{x^2}\,dx\\
\implies &v = 3\ln|x| + \frac{2}{x} + C\\
\implies &\frac{y}{x} = 3\ln|x| + \frac{2}{x} + C\\
\implies &y = 3x\ln|x| + 2 + Cx
\end{aligned}$$

(b) For stationary points, $\frac{dy}{dx} = 0$. $\therefore 0 = 3x + y - 2 \implies y = -3x + 2$.

(c)



3. (a)

$$\begin{aligned}
(x+y)^2 &= 2xy - \frac{2}{3}y^3\\
x^2 + 2xy + y^2 &= 2xy - \frac{2}{3}y^3\\
x^2 + y^2 &= -\frac{2}{3}y^3\\
2x + 2y\frac{dy}{dx} &= -2y^2\frac{dy}{dx}\\
y^2\frac{dy}{dx} + y\frac{dy}{dx} &= -x\\
(y^2 + y)\frac{dy}{dx} &= -x
\end{aligned}$$

(b)

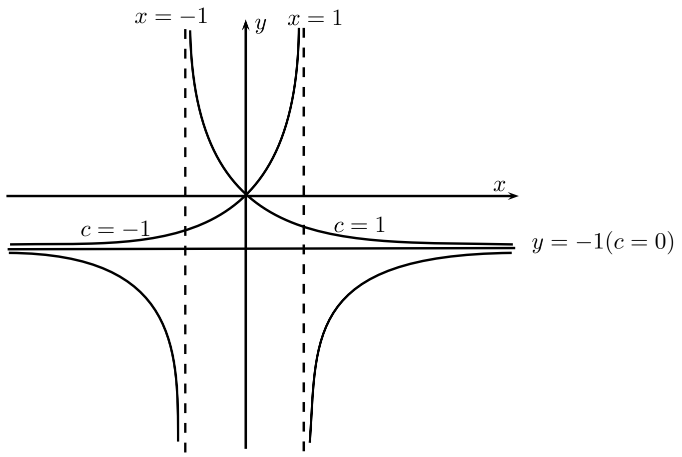
$$\begin{aligned}
y &= ux\\
\frac{dy}{dx} &= x\frac{du}{dx} + u
\end{aligned}$$

Substituting into the differential equation,

$$\begin{aligned} x\left(x\frac{du}{dx}+u\right) &= (ux)^2+ux \\ x\frac{du}{dx}+u &= u^2x+u \\ x\frac{du}{dx} &= u^2x \\ \frac{du}{dx} &= u^2 \\ \int \frac{1}{u^2}\,du &= \int 1\,dx \\ -\frac{1}{u} &= x+c \\ -\frac{x}{y} &= x+c \\ y &= -\frac{x}{x+c} \end{aligned}$$

(c) $y=-\frac{x}{x+c}=-1+\frac{c}{x+c}$

$c=-1, y=-1-\frac{1}{x-1}$
$c=0, y=-1$
$c=1, y=-1+\frac{1}{x+1}$



4. (a) $y=x \implies \frac{dy}{dx}=1.$

LHS $=1.$
RHS $=\frac{x^2+x^2}{2x^2}=1=$ RHS.

(b)

$$\begin{aligned} y &= ux \\ \frac{dy}{dx} &= x\frac{du}{dx}+u \end{aligned}$$

Substituting into the differential equation,

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2+y^2}{2xy} \\ x\frac{du}{dx}+u &= \frac{x^2+(ux)^2}{2x(ux)} \\ x\frac{du}{dx}+u &= \frac{1+u^2}{2u} \\ x\frac{du}{dx} &= \frac{1+u^2}{2u}-u \\ x\frac{du}{dx} &= \frac{1-u^2}{2u} \\ \int \frac{2u}{1-u^2}\,du &= \int \frac{1}{x}\,dx \\ \int \frac{-2u}{1-u^2}\,du &= -\int \frac{1}{x}\,dx \\ \ln|1-u^2| &= -\ln|x|+C \\ |1-u^2| &= e^{-\ln|x|+C} \\ |1-u^2| &= e^{\ln|x|^{-1}}e^C \\ |1-u^2| &= \frac{e^C}{|x|} \\ 1-u^2 &= \pm\frac{e^C}{x} \\ 1-u^2 &= A\left(\frac{1}{x}\right) \\ 1-\frac{y^2}{x^2} &= \frac{A}{x} \\ y^2 &= x^2-Ax \end{aligned}$$

(c) (i)

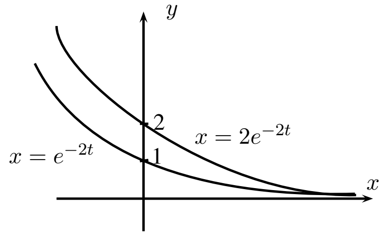
$$\begin{aligned} \frac{d^2x}{dt^2} &= 4ae^{-2t} \\ \frac{dx}{dt} &= -2ae^{-2t}+C \\ x &= ae^{-2t}+Ct+D \end{aligned}$$

Since the entire population is wiped out by the disease eventually, as $t\to\infty, x\to 0$.
Also, as $t\to\infty, e^{-2t}\to 0$. In order for $x\to 0$, we must have

$$C=0, D=0.$$

$\therefore x=ae^{-2t}$

(ii) a represents the initial population of the fish in thousands.



5. Let $y = u\sqrt{1-3x^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}\sqrt{1-3x^2} + u\left(\frac{1}{2}\right)\frac{-6x}{\sqrt{1-3x^2}}$

We start from DE : $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{3xy}{1-3x^2} - x + 1 = 0$

$$\frac{\mathrm{d}u}{\mathrm{d}x}\sqrt{1-3x^2} - \frac{-3xu}{\sqrt{1-3x^2}} + \frac{3x}{1-3x^2}(u\sqrt{1-3x^2}) - x + 1 = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}\sqrt{1-3x^2} - \frac{3xu}{\sqrt{1-3x^2}} + \frac{3xu}{\sqrt{1-3x^2}} = x - 1$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}\sqrt{1-3x^2} = x - 1$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{\sqrt{1-3x^2}} - \frac{1}{\sqrt{1-3x^2}}$$

$$u = -\frac{1}{6}\int \frac{-6x}{\sqrt{1-3x^2}}\ \mathrm{d}x - \int \frac{1}{\sqrt{1-3x^2}}\ \mathrm{d}x$$

$$\frac{y}{\sqrt{1-3x^2}} = -\frac{1}{6}[2\sqrt{1-3x^2}] - \frac{\sin^{-1}(\sqrt{3})}{\sqrt{3}} + C$$

$$y = -\frac{1}{3}(1-3x^2) - \frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}(\sqrt{3}x) + C\sqrt{1-3x^2}$$

6.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \sin x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \sec^2 x + \cos 2x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} = \int \sec^2 x + \cos 2x\ \mathrm{d}x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x + \frac{1}{2}\sin 2x + C$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x\tan x + \sin x + C\sec x$$

$$y = \int \sec x\tan x + \sin x + C\sec x\ \mathrm{d}x$$

$$y = \sec x - \cos x + C\ln|\sec x + \tan x| + D$$

