DIFFERENTIAL EQUATIONS

1 Solving Differential Equations

Separable Variables

$$\frac{dy}{dx} = f(y)g(x) \quad \Rightarrow \quad \frac{1}{f(y)}\frac{dy}{dx} = g(x)$$
$$\Rightarrow \quad \int \frac{1}{f(y)}\ dy = \int g(x)\ dx$$

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Example 1.
Find the general solution to the differential equation $\frac{dy}{dx} = \frac{2y-1}{x+2}$.

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Example 2.
Find the particular solution of the differential equation $y\frac{dy}{dx} + e^{-3y^2} = 0$, given that $y = 0$ when $x = 0$.
[$-\frac{1}{6}e^{3y^2} = x - \frac{1}{6}$]

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Example 3.
Find the general solution to the differential equation $x\frac{dy}{dx} = 1 + 2y$. $[y = \frac{Ax^2-1}{2}]$

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By Substitution

Step 1. Differentiate the given substitution equation wrt x.

Step 2. Replace $\frac{dy}{dx}$ and y accordingly with the expression in step 1.

Step 3. Solve the differential equation accordingly.

Step 4. Replace the variables to the original.

Example 4.
Using the substitution $v = y - x$, solve $\frac{dy}{dx} = \frac{1+y-x}{5+y-x}$.

Step 1. Differentiate the substitution.

$$\begin{aligned} v &= y - x \\ \frac{dv}{dx} &= \frac{dy}{dx} - 1 \\ \frac{dy}{dx} &= \frac{dv}{dx} + 1 \end{aligned}$$

Step 2. Replacement of variables.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1+y-x}{5+y-x} \\ \frac{dv}{dx} + 1 &= \frac{1+(v+x)-x}{5+(v+x)-x} \\ \frac{dv}{dx} + 1 &= \frac{1+v}{5+v} \\ \frac{dv}{dx} &= \frac{1+v}{5+v} - 1 \\ &= \frac{-4}{5+v} \end{aligned}$$

Step 3. Solve the DE.

$$\begin{aligned} \frac{dv}{dx} &= \frac{-4}{5+v} \\ \frac{5+v}{-4}\frac{dv}{dx} &= 1 \\ \int \frac{5+v}{-4}\ dv &= \int 1\ dx \\ -\frac{1}{4}\left(5v + \frac{v^2}{2}\right) &= x + C \end{aligned}$$

Step 4. Replacement of variables.

$$-\frac{1}{4}\left(5(y-x) + \frac{(y-x)^2}{2}\right) = x + C$$

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Example 5.

Find the general solution of the following differential equation by means of the suggested substitution:

$$x\frac{dy}{dx} - y = x(x-y); \quad y = vx$$

$[y = x - Axe^{-x}]$

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Example 6 (2014/JJC/Prelim/II/2a)**.**

Find the general solution of the differential equation $\frac{d^2y}{dx^2} = ae^{-2x}$, where a is a constant.

Solution:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= ae^{-2x} \\
\int \frac{d^2y}{dx^2}\, dx &= \int ae^{-2x}\, dx \\
\frac{dy}{dx} &= \frac{ae^{-2x}}{-2} + C \\
\int \frac{dy}{dx}\, dx &= \int \left(\frac{ae^{-2x}}{-2} + C\right)\, dx \\
y &= \frac{ae^{-2x}}{4} + Cx + d
\end{aligned}$$

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2 Family of Solution Curves

Suppose we have the differential equation

$$\frac{dy}{dx} = 2x.$$

The general solution to the DE is

$$y = x^2 + C.$$

This value of C depends on a particular coordinate that the curve passes through.

Value of C	Equation of solution
$C = -1$	$y = x^2 - 1$
$C = 0$	$y = x^2$
$C = 1$	$y = x^2 + 1$

Sketching these 3 curves on a graph, we obtain a family of solution curves,

$$y = x^2 + C, \qquad \text{for } C = -1, 0, 1.$$

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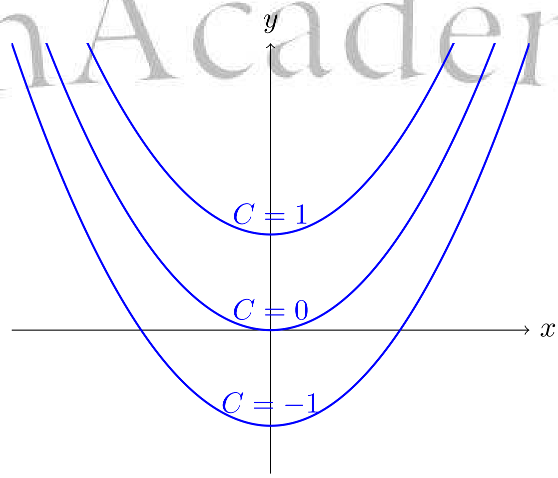

Sketching family of solution

When asked to sketch the family of solution curves, we will sketch for 3 values,

(i) $C < 0$,

(ii) $C = 0$,

(iii) $C > 0$.

For convenience, we usually sketch for $C = -1, 0, 1$.

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Example 7.
[2011/RI/Prelim/I/8]

(a) Given that y is a function of x, find $\frac{d}{dx}(ye^{3x})$ in terms of x, y and $\frac{dy}{dx}$. [1]

(b) Hence, or otherwise, find the general solution of $\frac{dy}{dx} + 3y = 1$, expressing y in terms of x. [2]

(c) Draw and label clearly the family of curves of your general solution to (b). [3]

Solution:

(a) $\frac{d}{dx}(ye^{3x}) = y(3e^{3x}) + e^{3x}\frac{dy}{dx} = 3ye^{3x} + \frac{dy}{dx}e^{3x}$

(b) $\frac{dy}{dx} + 3y = 1$, multiplying both sides by e^{3x},

$$\begin{aligned}
\frac{dy}{dx} + 3y &= 1 \\
\frac{dy}{dx}e^{3x} + 3ye^{3x} &= e^{3x} \\
\int \left(\frac{dy}{dx}e^{3x} + 3ye^{3x}\right) dx &= \int e^{3x}\, dx \\
\int \frac{d}{dx}(ye^{3x})\, dx &= \int e^{3x}\, dx \\
ye^{3x} &= \frac{e^{3x}}{3} + C \\
y &= \frac{1}{3} + Ce^{-3x}, \quad \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

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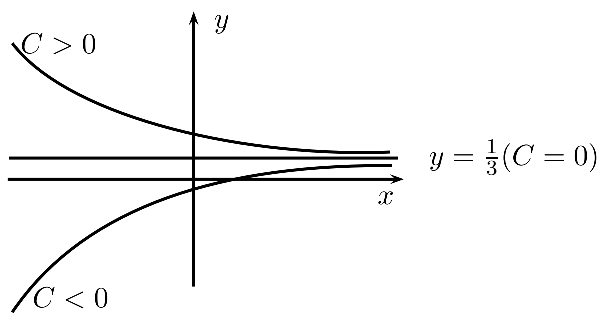
(c)



*Suppose the question asks you to sketch the non-linear members of the family of curves, what would be different?

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3 Applications of Differential Equations

Forming DE

Let y be a function of x.

$\frac{dy}{dx} > 0 \quad \Rightarrow \quad y$ **increases** as x increases
$\frac{dy}{dx} < 0 \quad \Rightarrow \quad y$ **decreases** as x increases

Given that y is positive,

(a) Rate of increase of x is proportional to y,

$$\frac{dx}{dt} \propto y \qquad \Longrightarrow \qquad \frac{dx}{dt} = ky, \quad k > 0$$

(b) Rate of decrease of x is proportional to y,

$$\frac{dx}{dt} \propto -y \qquad \Longrightarrow \qquad \frac{dx}{dt} = -ky, \quad k > 0$$

Example 8 (2010/RVHS/Prelim/II/4a)**.**
A long cyclindrical metal bar is submerged into iced water. A researcher claims that the rate at which the length, l cm, of the bar is shrinking at any time t seconds is proportional to the volume of the bar at that instant, assuming that the cross-sectional areas of the bar remains constant during the shrinking process. Formulate and integrate a differential equation to show that $l = Ae^{kt}$, where A and k are constants. State the range of values of A and of k. [5]

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Solution:

Let D be the cross section area.

$$\begin{aligned}
\frac{dl}{dt} &\propto -V \\
\frac{dl}{dt} &\propto -Dl \\
\Longrightarrow \frac{dl}{dt} &= BDl, \text{ (for some constant } B < 0) \\
\Longrightarrow \frac{dl}{dt} &= kl, \text{ (where } k = BD < 0) \\
\int \frac{1}{l}\, dl &= \int k\, dt \\
\ln l &= kt + C \\
l &= e^{kt+C} \\
l &= Ae^{kt}, A > 0 \text{ (shown)}
\end{aligned}$$

Range of A and k: $A > 0, k < 0$.

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Example 9 (2013/MJC/Prelim/I/7).
A tank contains 2 m^3 of water initially. Water flows into the tank at a constant rate of 4 m^3s^{-1} and flows out at a rate which is proportional to the amount of water V m^3 in the tank. The volume of water in the tank remains unchanged at the instant when $V = 8$.

(i) By setting up and solving a differential equation, show that $V = 8 - 6e^{-0.5t}$.

(ii) What will happen to the volume of water in the tank eventually?

(i)

$\frac{dV}{dt} = 4 - kV$

When $V = 8, \frac{dV}{dt} = 0$,
$\therefore 0 = 4 - 8k$

$\implies k = \frac{1}{2}$

$\therefore \frac{dV}{dt} = 4 - \frac{1}{2}V$

$$\int \frac{1}{4 - \frac{1}{2}V}\, dV = \int 1\, dt$$

$$-2\int \frac{-\frac{1}{2}}{4 - \frac{1}{2}V}\, dV = t + c$$

$$-2\ln\left|4 - \frac{1}{2}V\right| = t + c$$

When $t = 0, V = 2$,

$$-2\ln\left|4 - \frac{1}{2}(2)\right| = c$$

$$c = -2\ln 3$$

$$\therefore -2\ln\left|4 - \frac{1}{2}V\right| = t - 2\ln 3$$

$$\ln\left|4 - \frac{1}{2}V\right| = -\frac{1}{2}t + \ln 3$$

$$\left|4 - \frac{1}{2}V\right| = e^{-\frac{1}{2}t} \cdot e^{\ln 3}$$

$$\left|4 - \frac{1}{2}V\right| = 3e^{-0.5t}$$

Note that volume of water is at most 8 m^3.
$\therefore \left|4 - \frac{1}{2}V\right| = 4 - \frac{1}{2}V$

$$4 - \frac{1}{2}V = 3e^{-0.5t}$$

$$V = 8 - 6e^{-0.5t}$$

(ii) As $t \to \infty, e^{-0.5t} \to 0. \therefore V \to 8$

Volume of water will increase and converge to 8 m^3.

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