AP GP

1 Arithmetic Progression

$$\begin{array}{llllll} 1, & 6, & 11, & 16, & 21, & \dots \\ 1, & 1+5, & 1+(2)5, & 1+(3)5, & 1+(4)5, & \dots \\ a, & a+d, & a+2d, & a+3d, & a+4d, & \dots \end{array}$$

first term : a
common difference : d

A sequence of the above form is called an **arithmetic progression**, or A.P.

n^{th} term forumla

$$T_n = a + (n-1)d$$

Example 1.
In an arithmetic progression, the 6^{th} term is twice the 3^{rd} term and the 10^{th} term is 20. Find the common difference in this sequence.

Solution:
$T_6 = a + 5d$.
$T_3 = a + 2d$.
Given that $T_6 = 2 \times T_3$,

$$\begin{aligned} a + 5d &= 2(a + 2d) \\ a + 5d &= 2a + 4d \\ d &= a \quad \cdots (1) \end{aligned}$$

Given that the 10^{th} term is 20,

$$a + 9d = 20 \quad \cdots (2)$$

Solving simultaneous equation with (1) and (2), we get $d = 2$.

Example 2.
Given that the 4^{th} and 8^{th} term of an A.P. are 7 and 35 respectively, find the n^{th} term of the sequence.
$[a = -14, d = 7; 7n - 21]$

Example 3.
Find the number of terms in the arithmetic progression $5, 7, 9, \ldots, 97$. $[n = 47]$

2 Sum of Arithmetic Progression

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

Sum of A.P.

The sum of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}(a + \ell).$$

n: number of terms
a: first term
ℓ: last term (n^{th} term)

Since $\ell = a + (n-1)d$,

$$S_n = \frac{n}{2}\left[a \quad + \quad a + (n-1)d\right]$$

where d is the common difference.

Example 4 (2007/MJC/P1/1)**.**
An arithmetic progression G has first term a and a non-zero common difference d. Given that the 6^{th} term is 37 and the sum of the first 10 terms in G is twice the 22^{nd} term of the series, find the values of a and d. [3]
A new series H is formed by selecting every third term of G. Find the sum of the first 50 terms of H. [3]

Solution:

$$\begin{aligned} 6^{th} \text{ term} &= 37 \\ a + 5d &= 37 \quad \cdots (1) \end{aligned}$$

$$\begin{aligned} \text{Sum of first 10 terms in } G &= 2 \times 22^{nd} \text{ term} \\ \frac{10}{2}[a + (a + 9d)] &= 2(a + 21d) \\ &\vdots \\ a &= -\frac{3}{8}d \quad \cdots (2) \end{aligned}$$

Solving the simultaneous equation using (1) and (2), we get:

$$d = 8, \quad a = -3.$$

$$\begin{aligned} &\text{Sum of first 50 terms in } H \\ =&(a + 2d) + (a + 5d) + (a + 8d) + \cdots + 50^{th} \text{ term} \\ =&\text{ Sum of A.P. with first term: } a + 2d = 13 \text{ and common difference: } 3d = 24 \\ =&\frac{50}{2}[13 \quad + \quad 13 + (50-1)(24)] \\ =&30050 \end{aligned}$$

Example 5.

Find the sum of all the odd numbers from 3 to 99 inclusive which are not multiples of 5.
(Hint: Find separately, the sum of the odd numbers AND those that are multiples of 5.)

[1999]

Example 6 (2012/MI/Prelim/I/5).
The 9^{th} term of an arithmetic progression is 50 and the sum of the first 15 terms is 570. It is given that the sum of the first n terms is greater than 500. Find the least possible value of n.

[15]

$$S_n = T_1 + T_2 + \cdots + T_{n-1} + T_n$$
$$S_{n-1} = T_1 + T_2 + \cdots + T_{n-1}$$

Finding T_n fron S_n

$$T_n = S_n - S_{n-1}$$

Example 7.
The sum of the first n terms of a series is given by $S_n = n(4+n)$. Find the general formula for the n^{th} term.

Solution:

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= n(4+n) - (n-1)(4+n-1) \\ &= 4n + n^2 - (n-1)(n+3) \\ &= 4n + n^2 - (n^2 + 2n - 3) \\ &= 4n + n^2 - n^2 - 2n + 3 \\ &= 2n + 3 \end{aligned}$$

Prove that a sequence is an A.P.

We show that the common difference is a constant, i.e,

$$T_n - T_{n-1} = d$$

Example 8.
The sum to n terms of a series is $S_n = 8n^2 - 4n$. Calculate the n^{th} term. Hence prove that the sequence is an arithmetic progression.

Solution:

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 8n^2 - 4n - [8(n-1)^2 - 4(n-1)] \\ &\vdots \\ &= 16n - 12 \end{aligned}$$

$$\begin{aligned} T_n - T_{n-1} &= 16n - 12 - [16(n-1) - 12] \\ &= 16n - 12 - 16n + 16 + 12 \\ &= 16 \end{aligned}$$

Since $T_n - T_{n-1}$ is a constant, the sequence is an A.P.

3 Geometric Progression

$$2, \quad 2(3), \quad 2(3)^2, \quad 2(3)^3, \quad 2(3)^4, \ldots$$

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \ldots$$

$$\begin{aligned} \text{first term :} \quad & a \\ \text{common ratio :} \quad & r \\ n^{th} \text{ term: :} \quad & T_n = ar^{n-1} \end{aligned}$$

A sequence of the above form is called a **geometric progression**, or G.P.

3.1 Properties of G.P.

Summary of Problem Solving Tools

(i) Sum of first n terms,

$$S_n = \frac{a(1-r^n)}{1-r}.$$

(ii) Sum of infinite terms,

$$S_\infty = \frac{a}{1-r}.$$

(iii) Condition for S_∞ to exist,

$$|r| < 1.$$

(iv) If x, y and z are consecutive terms in a G.P. Then,

$$\frac{y}{x} = \frac{z}{y}. \qquad \text{(useful to solve problems)}$$

(v) To show that a sequence is a G.P., we show

$$\frac{T_{n+1}}{T_n} = \text{constant}.$$

Sum of G.P.

(a) Sum of first n terms,

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

(b) Sum of infinite terms (only for $|r| < 1$),

$$S_\infty = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Let $a = 3$ and $r = 0.1$,

$$\begin{aligned} & 3 + 0.3 + 0.03 + 0.003 + 0.0003 + \ldots \\ =& 3.3333\ldots \\ =& 3\frac{1}{3} \end{aligned}$$

The series is convergent.

Let $a = 1$ and $r = 10$,

$$\begin{aligned} & 1 + 10 + 10^2 + 10^3 + 10^4 + 10^5 \ldots \\ =& 11111\ldots \end{aligned}$$

The series is divergent.

When $|r| < 1$, the infinite geometric series is **convergent**.
When $|r| \geq 1$, the infinite geometric series is **divergent**.

n^{th} **term formula**

$$T_n = ar^{n-1}$$

Example 9.
The second term of a geometric progression is 5 and its sum to infinity is 20. Find the sum of the first 4 terms.

Solution:

$$\begin{aligned} T_2 &= 5 \\ ar &= 5 \end{aligned}$$

$$\begin{aligned} S_\infty &= 20 \\ \frac{a}{1-r} &= 20 \end{aligned}$$

Solving simultaneous equation, we get, $r = \frac{1}{2}$, $a = 10$.

$$S_4 = \frac{a(1-r^4)}{1-r} = \frac{10[1-(\frac{1}{2})^4]}{1-\frac{1}{2}} = \frac{75}{4}$$

Condition for series of G.P. to be convergent

$$|r| < 1.$$

Example 10.
A geometric series has first term a. Find the range of values of a if the sum to infinity of the series is $\frac{1}{2}$. [DHS/I/Q10]

Solution:

$$\begin{aligned} \frac{a}{1-r} &= \frac{1}{2} \\ 2a &= 1-r \\ r &= 1-2a \end{aligned}$$

Since the sum to infinity exists, $|r| < 1$.

$$\begin{aligned} |r| &< 1 \\ |1-2a| &< 1 \\ -1 < 1-2a &< 1 \end{aligned}$$

$$\begin{aligned} -1 &< 1-2a \\ a &< 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 1-2a &< 1 \\ 0 &< 2a \\ a &> 0 \end{aligned}$$

$$0 < a < 1$$

Terms form a G.P.

If x, y and z are consecutive terms in a GP. Then,

$$\frac{y}{x} = \frac{z}{y}.$$

Prove G.P. is convergent.

We first find the common ratio r, then we show that,

$$|r| < 1.$$

Example 11 (Terms of AP forms a GP).
[2011/IJC/I/7]

An arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of a geometric series are equal to the ninth, fifth and second terms respectively of the arithmetic series.

(a) Prove that the geometric series is convergent and find, in terms of a, the sum to infinity. [5]

(b) Given that $a > 0$, find the largest value of n for which the sum of the first n terms of the geometric series is less than four fifths of the sum to infinity. [5]

[(a)$8a$, (b)5]

Solution:

For the A.P.,

$$9^{th} \text{ term:} = a + 8d$$
$$5^{th} \text{ term:} = a + 4d$$
$$2^{nd} \text{ term:} = a + d$$

Since the terms form the first 3 terms of a G.P.,

$$\begin{aligned}
\frac{a+4d}{a+8d} &= \frac{a+d}{a+4d} \\
(a+4d)^2 &= (a+8d)(a+d) \\
a^2 + 8ad + 16d^2 &= a^2 + 9ad + 8d^2 \\
8d^2 - ad &= 0 \\
d(8d - a) &= 0 \\
d = 0 \quad &\text{or} \quad d = \frac{a}{8} \\
(\text{rej as } d \neq 0) &
\end{aligned}$$

(a) Therefore, common ratio

$$r = \frac{a+4d}{a+8d} = \frac{a+\frac{4a}{8}}{a+\frac{8a}{8}} = \frac{1.5a}{2a} = \frac{3}{4} < 1.$$

Hence the series is convergent.

First term of the geometric series $= a + 8(\frac{a}{8}) = 2a$

$$S_\infty = \frac{2a}{1-\frac{3}{4}} = \frac{2a}{\frac{1}{4}} = 8a.$$

(b)

$$S_n < \frac{4}{5}S_\infty$$

$$\frac{2a\left[1-\left(\frac{3}{4}\right)^n\right]}{1-\frac{3}{4}} < \frac{4}{5}(8a)$$

$$8a\left[1-\left(\frac{3}{4}\right)^n\right] < \frac{4}{5}(8a)$$

$$1-\left(\frac{3}{4}\right)^n < \frac{4}{5} \qquad \text{(Since } a > 0\text{)}$$

$$\frac{1}{5} < \left(\frac{3}{4}\right)^n$$

$$\ln\frac{1}{5} < n\ln\left(\frac{3}{4}\right)$$

$$\frac{\ln\frac{1}{5}}{\ln\left(\frac{3}{4}\right)} > n \qquad \text{(Since } \ln\tfrac{3}{4} < 0\text{, the sign changes direction)}$$

$$5.59 > n$$

Therefore, largest $n = 5$.

Show a sequence is a G.P.

If we are given S_n, we will first find T_n through

$$T_n = S_n - S_{n-1}.$$

Then, we show

$$\frac{T_{n+1}}{T_n} = \text{constant}.$$

Example 12.
The sum of the first n terms of a series is $1-\left(\frac{1}{3}\right)^n$. Show that the terms of this series are in a geometric progression. State the first term and the common ratio.

Solution:

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= \left[1-\left(\frac{1}{3}\right)^n\right] - \left[1-\left(\frac{1}{3}\right)^{n-1}\right] \\ &= \left(\frac{1}{3}\right)^{n-1} - \left(\frac{1}{3}\right)^n \\ &= \left(\frac{1}{3}\right)^{n-1}\left(1-\frac{1}{3}\right) \\ &= \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} \end{aligned}$$

$$\frac{T_{n+1}}{T_n} = \frac{\frac{2}{3}\left(\frac{1}{3}\right)^n}{\frac{2}{3}\left(\frac{1}{3}\right)^{n-1}} = \frac{1}{3}.$$

Since $\frac{T_{n+1}}{T_n}$ is a constant, it is a G.P. with first term $= \frac{2}{3}$ and common ratio $= \frac{1}{3}$.

4 Practical Situations Involving APGP

4.1 Financial Situations

Example 13 (2012/MI/Prelim/I/5a)**.**
An educational fund is started at \$2000 and the bank offers a compound interest at 2% per annum, at the last day of each year. If withdrawals of \$50 are made at the beginning of each of the subsequent years, show that the amount in the fund at the beginning of the $(n+1)^{th}$ year is \$500(5-1.02^n).

Solution:

Year	Beginning of the year	End of the year

Beginning of $n+1$ year:

$$\begin{aligned}
&2000(1.02)^n - 50(1.02)^{n-1} - 50(1.02)^{n-2} - \ldots - 50(1.02) - 50 \\
=&2000(1.02)^n - 50[1.02^{n-1} + 1.02^{n-2} + \ldots + 1.02 + 1] \\
=&2000(1.02)^n - 50\left(\frac{1-1.02^n}{1-1.02}\right) \\
=&2000(1.02)^n + 2500(1-1.02^n) \\
=&2500 - 500(1.02^n) \\
=&500(5-1.02^n)
\end{aligned}$$

Example 14 (2018/PJC/Prelim/I/9bi).

A man takes a loan of \$$P$ for a house from a bank at the beginning of a month. The interest rate is 0.5% per month so that at the start of every month, the amount of outstanding loan is increased by 0.5%. Equal instalment is paid to the bank at the end of every month. Find his monthly instalment if he would like to repay the loan in 20 years, leaving your answer in the form $0.005P\frac{k^n}{k^n-1}$, where k and n are constants to be determined.

Year	Beginning of the month	End of the month

Example 15 (2015/VJC/Prelim/I/6a).
On 1 January 2015, Mrs Koh put \$1000 into an investment fund which pays compound interest at a rate of 8% per annum on the last day of each year. She puts a further \$1000 into the fund on the first day of each subsequent year until she retires.

If she retires on 31 December 2040, show that the total value of her investment on her retirement day is \$86351, correct to the nearest dollar. [4]

Example 16 (2015/VJC/Prelim/I/6b).
On 1 January 2015, Mr Woo put \$1000 into a savings plan that pays no interest. On the first day of each subsequent year, he saves \$80 more than the previous year. Thus, he saves \$1080 on 1 January 2016, \$1160 on 1 January 2017, and so on.

By forming a suitable inequality, find the year in which Mr Woo will first have saved over S\$86351 in total. [4] [Year 2050]

4.2 Non-Financial Situations and Limits

For a sequence of real numbers $x_1, x_2, x_3, \ldots,$

$$\text{when } n \longrightarrow \infty, \quad \text{if } x_n \longrightarrow s,$$

then the sequence is said to be convergent (to s) and s is called the **limit** of the sequence.

We represent the limit as

$$\lim_{n\to\infty} x_n = s.$$

Condition	Variable	Example
Nil	$\lim_{n\to\infty} \frac{1}{n} = 0$	$\lim_{n\to\infty} \frac{1}{3+2n} = 0$
$\lvert a\rvert < 1$	$\lim_{n\to\infty} a^n = 0$	$\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$
$b > 1$	As $n \to \infty$, $b^n \to \infty$	As $n \to \infty$, $3^n \to \infty$

Example 17.
The volume of water, in litres, in a storage tank decreases by 10% at the start of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of n days is denoted by x_n and x_0 is the initial volume of water in the tank. It is also known that $x_0 > 900$.

(i) Show that $x_n = 0.9^n(x_0 - 900) + 900$.

(ii) Explain why the volume of water will never fall below 900 litres.

Solution:

x_n	Start of the day	End of the day
x_1	$0.9x_0$	$0.9x_0 + 90$
x_2	$0.9\left[0.9x_0 + 90\right]$ $= 0.9^2x_0 + 0.9(90)$	$0.9^2x_0 + 0.9(90) + 90$
x_3	$0.9\left[(0.9)^2x_0 + 0.9(90) + 90\right]$ $= 0.9^3x_0 + 0.9^2(90) + 0.9(90)$	$0.9^3x_0 + 0.9^2(90) + 0.9(90) + 90$

(i)

$$\therefore x_n = 0.9^n x_0 + \underbrace{0.9^{n-1}(90) + 0.9^{n-2}(90) + \cdots + 0.9^2(90) + 0.9(90) + 90}_{\substack{\text{GP with first term} = 90 \\ \text{common ratio} = 0.9 \\ \text{number of terms} = n}}$$

$$\begin{aligned} &= 0.9^n x_0 + \frac{90(1-0.9^n)}{1-0.9} \\ &= 0.9^n x_0 + 900(1-0.9^n) \\ &= 0.9^n x_0 + 900 - 0.9^n(900) \\ &= 0.9^n(x_0 - 900) + 900 \end{aligned}$$

(ii)

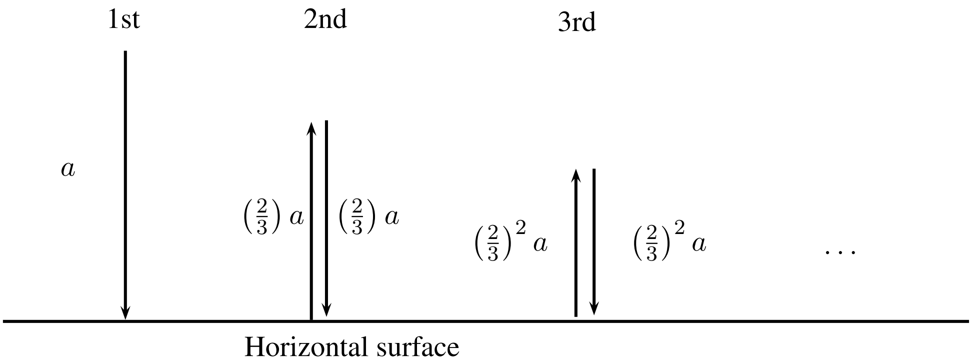
$$\lim_{n\to\infty} x_n = 900$$

Since x_n is a decreasing sequence and its limit is 900, the volume of water will not fall below 900 litres.

Example 18 (2014/YJC/Prelim/I/10b).
A ball is dropped from a height of a metres above a horizontal surface and it rebounds to two-thirds of its previous height,

(i) Show that the total distance that the ball has travelled at the instant when it hits the surface for the nth time is $5a - 4a\left(\frac{2}{3}\right)^{n-1}$. [3]

(ii) If L is the total distance travelled by the ball until it stops bouncing, write down the value of L in terms of a. [1]



Solution:

(i)

$$\begin{aligned}\text{Total distance} &= a + 2\left(\frac{2}{3}\right)a + 2\left(\frac{2}{3}\right)^2 a + \ldots + 2\left(\frac{2}{3}\right)^{n-1} a \\ &= a + 2a\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots \left(\frac{2}{3}\right)^{n-1}\right] \\ &= a + 2a\left[\frac{\frac{2}{3}\left(1 - \left(\frac{2}{3}\right)^{n-1}\right)}{1 - \frac{2}{3}}\right] \\ &= a + 4a\left[1 - \left(\frac{2}{3}\right)^{n-1}\right] \\ &= a + 4a - 4a\left(\frac{2}{3}\right)^{n-1} \\ &= 5a - 4a\left(\frac{2}{3}\right)^{n-1}\end{aligned}$$

(ii) As $n \to \infty$, $\left(\frac{2}{3}\right)^{n-1} \to 0$, hence, total distance $\to 5a$. $\therefore L = 5a$.

