APGP Tutorial 1: Solutions

1. Strategy: We will find the sum of all terms from 1102 to 2011, subtract by sum all the terms that are divisible by 3.

   From 1102 to 2011,

$$\begin{aligned} \text{First term that is divisible by 3:} &\quad 1104 \\ \text{Last term that is divisible by 3:} &\quad 2010 \end{aligned}$$

   Note that the terms

$$1104, 1107, 1110, \ldots, 2007, 2010$$

   forms an AP with 1st term 1104, common difference 3. First, find the number of terms that are divisible by 3.

$$\begin{aligned} a + (n-1)d &= 2010 \\ 1103 + (n-1)(3) &= 2010 \\ n &= 303 \end{aligned}$$

   Sum of terms divisible by 3: $\frac{303}{2}(1104 + 2010) = 471771$

   Number of terms from 1102 to 2011: $2011 - 1102 + 1 = 910$
   Sum of all terms: $\frac{910}{2}(1102 + 2011) = 1416415$

   Sum required: 1416415 - 471771 = 944644

2.

$$\begin{aligned} S_{10} &= 105 + 10u_5 \\ \frac{10}{2}(a + a + 9d) &= 105 + 10(a + 4d) \\ 5(2a + 9d) &= 105 + 10a + 40d \\ 10a + 45d &= 105 + 10a + 40d \\ 5d &= 105 \\ d &= 21 \end{aligned}$$

$$\begin{aligned} u_{26} &> 542 \\ a + 25d &> 542 \\ a + 25(21) &> 542 \\ a &> 17 \end{aligned}$$

$$\begin{aligned} u_{25} &\leq 542 \\ a + 24d &\leq 542 \\ a + 24(21) &\leq 542 \\ a &\leq 38 \end{aligned}$$

   $\therefore 17 < a \leq 38.$

3.

$$\begin{aligned}\text{Sum of last 5 terms} &= u_{n-4}+u_{n-3}+u_{n-2}+u_{n-1}+u_n \\ &= \frac{5}{2}(u_{n-4}+u_n) \\ &= \frac{5}{2}[1+(n-5)d+1+(n-1)d] \\ &= \frac{5}{2}[2+2nd-6d] \\ &= 5+5nd-15d\end{aligned}$$

$$\begin{aligned}\text{Sum of first 4 terms} &= \frac{4}{2}(1+1+3d) \\ &= 2(2+3d) \\ &= 4+6d\end{aligned}$$

$$\begin{aligned}\text{Sum of last 5 terms} &= \text{Sum of first 4 terms} + 193 \\ 5+5nd-15d &= 4+6d+193 \\ 5nd-21d-192 &= 0 \text{ (shown)}\end{aligned}$$

$$\begin{aligned}6^{th}\text{ term} &= 16 \\ 1+5d &= 16 \\ 5d &= 15 \\ d &= 3\end{aligned}$$

$$\begin{aligned}\therefore 5n(3)-21(3)-192 &= 0 \\ n &= 17\end{aligned}$$

4.

$$\begin{aligned}T_n &= S_n - S_{n-1} \\ &= 2n^2+kn-3-[2(n-1)^2+k(n-1)-3] \\ &= 2n^2+kn-3-[2n^2-4n+2+kn-k-3] \\ &= 2n^2+kn-3-[2n^2+nk-4n-k-1] \\ &= 4n+k-2\end{aligned}$$

$$\begin{aligned}T_n - T_{n-1} &= 4n+k-2-[4(n-1)+k-2] \\ &= 4n+k-2-4n+4-k+2 \\ &= 4\end{aligned}$$

Since $T_n - T_{n-1}$ is a constant, the sequence is an A.P.

5.

$$\begin{aligned}u_n &= S_n - S_{n-1} \\ &= \pi\left(n-\frac{1}{2}\right)^2 - \frac{\pi}{4} + n\pi^2 - \pi\left((n-1)-\frac{1}{2}\right)^2 + \frac{\pi}{4} - (n-1)\pi^2 \\ &= \pi\left(n^2-n+\frac{1}{4}\right) - \frac{\pi}{4} + n\pi^2 - \pi\left(n^2-3n+\frac{9}{4}\right) + \frac{\pi}{4} - n\pi^2 + \pi^2 \\ &= -n\pi + \frac{1}{4}\pi + 3n\pi - \frac{9}{4}\pi + \pi^2 \\ &= 2n\pi - 2\pi + \pi^2\end{aligned}$$

To show AP,

$u_n - u_{n-1} = 2n\pi - 2\pi + \pi^2 - \left(2(n-1)\pi - 2\pi + \pi^2\right) = 2\pi$

Since $u_n - u_{n-1}$ is a constant, sequences follows an AP with common difference 2π.

6. (a)

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 3n^2 - 3(n-1)^2 \\ &= 3n^2 - 3(n^2 - 2n + 1) \\ &= 3n^2 - 3n^2 + 6n - 3 \\ &= 6n - 3 \end{aligned}$$

$$\begin{aligned} T_n - T_{n-1} &= 6n - 3 - [6(n-1) - 3] \\ &= 6n - 3 - (6n - 9) \\ &= -3 + 9 \\ &= 6 \end{aligned}$$

Since $T_n - T_{n-1}$ is a constant, the sequence is an A.P.

(b)

$$\begin{aligned} 3n^2 &> 244 \\ n^2 &> 81.333 \\ (n - 9.018)(n + 9.018) &> 0 \\ n > 9.018 \text{ or } n &< -9.018(\text{rej}) \end{aligned}$$

$$\therefore \text{least } n = 10.$$

7.

$$\begin{aligned} \frac{n}{2}\left(5 + 5 + (n-1)(6)\right) &= 2760 \\ 3n^2 + 2n - 2760 &= 0 \\ n = 30 \text{ or } n &= \frac{-92}{3}(\text{rej as } n > 0) \end{aligned}$$

$\therefore x = T_{30} = 5 + 29(6) = 179$

8. Let n be the number of boxes that Adam requires.

$$\begin{aligned} \text{Total number of all marbles in } n \text{ boxes} &= 6643 \\ \frac{n}{2}\left[13 + 13 + (n-1)(13)\right] &= 6643 \\ \frac{n}{2}\left[13n + 13\right] &= 6643 \\ 13n^2 + 13n - 13286 &= 0 \\ n = 31.473 \text{ or } &- 32.473 \end{aligned}$$

Hence, he will required 32 boxes.

We now count how many marbes there are in the first 31 boxes.

$$\frac{31}{2}\left(13 + 13 + 30(13)\right) = 6448$$

Therefore, number of marbles in the last box is 6643 - 6448 = 195.

9. (a) 1st set: 1 term
2nd set: 2 terms

3rd set: 3 terms
⋮
nth set: n terms

Total number of terms $= 1 + 2 + 3 + \cdots + n = \frac{n}{2}(1+n)$

(b)

Set	Number of Terms so far	First term of the Set	Last term of the set
1	1	T_1	T_1
2	1+2 = 3	T_2	T_3
3	1+2 +3 = 6	T_4	T_6
4	1+2 +3 +4 =10	T_7	T_{10}
5	1+2 +3 +4 +5=15	T_{11}	T_{15}
$n-1$			
n			

Using the results from (a), total number of terms in the first $(n-1)$ sets $= \frac{n-1}{2}(n)$. Therefore, the term corresponding to the first term in the n^{th} set is the $\left(\frac{n-1}{2}(n)+1\right)^{th}$ term.

$$\begin{aligned} T_{\frac{n-1}{2}(n)+1} &= 2 + \left[\left(\frac{n-1}{2}(n)+1\right) - 1\right](4) \qquad \text{(Using AP formula } T_n = a + (n-1)d) \\ &= 2n^2 - 2n + 2 \end{aligned}$$

(c) Total number of terms in the first n sets $= \frac{n}{2}(1+n)$. Therefore, the term corresponding to the last term in the n^{th} set is the $\left(\frac{n}{2}(1+n)\right)^{th}$ term.

$$\begin{aligned} T_{\frac{n}{2}(1+n)} &= 2 + \left[\frac{n}{2}(1+n) - 1\right](4) \qquad \text{(Using AP formula } T_n = a + (n-1)d) \\ &= 2n^2 + 2n - 2 \end{aligned}$$

(d) First term in n^{th} set: $2n^2 - 2n + 2$
Last term in n^{th} set: $2n^2 + 2n - 2$

$$\text{Sum of all terms in } n^{th} \text{ set} = \frac{n}{2}[(2n^2 - 2n + 2) + 2n^2 + 2n - 2] = 2n^3$$