COMPLEX NUMBERS

1 Complex Number Basics

1.1 Basic operations

Definition

Let $i = \sqrt{-1}$. A complex number is a number of the form

$$z = x + iy, \quad \text{where } x, y \in \mathbb{R}$$

x is the real part of z, denoted by $\mathrm{Re}(z)$.
y is the imaginary part of z, denoted by $\mathrm{Im}(z)$.

Powers of i

i^0	1
i	i
i^2	-1
i^3	$-i$

i^4	1
i^5	i
i^6	-1
i^7	$-i$

The pattern $1, i, -1, -i, \ldots$ will repeat itself.

Complex conjugate z^*

If $z = x + iy$, then its *complex conjugate* z^* is defined as

$$z^* = x - iy$$

Observe the following:

$$zz^* = x^2 + y^2$$

Thus, the product of a complex number and its complex conjugate is a real number.

Proof:

$$\begin{aligned} zz^* &= (x+yi)(x-yi) \\ &= x^2 - (yi)^2 \\ &= x^2 - (-y^2) \\ &= x^2 + y^2 \end{aligned}$$

Operations of Complex Numbers

(i) **Equality**

$$a + bi = c + di \quad \Longleftrightarrow a = c \quad \text{AND} \quad b = d$$

(ii) **Addition/Subtraction**

$$\begin{aligned} (a+bi) + (c+di) &= (a+c) + (b+d)i \\ (a+bi) - (c+di) &= (a-c) + (b-d)i \end{aligned}$$

(iii) **Multiplication**

$$\begin{aligned} (a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

(iv) **Division** (Multiply denominator by its complex conjugate)

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{(a+bi)(c-di)}{c^2+d^2} \end{aligned}$$

Example 1.
Express the following in the form $x+iy$, where $x,y\in\mathbb{R}$. [(b) $1+i(-\sqrt{2}+3)$ (d)7 (f)$\frac{10}{17}+\frac{11}{17}i$]

a) $z=(3-3i)+(-3+i)$

b) $z=(2-\sqrt{2}i)-(1-3i)$

c) $z=(2+2i)(1-3i)$

d) $z=(\sqrt{3}+2i)(\sqrt{3}-2i)$

e) $z=\frac{2+10i}{5-3i}$

f) $z=\frac{3+2i}{4-i}$

1.2 Solving Complex Equations

Example 2.
Given that $(2-i)^2+(3\lambda+i)(\mu-i)+5i=10$, find the exact values of λ and μ.

Solution:

$$\begin{aligned}(2-i)^2+(3\lambda+i)(\mu-i)+5i&=10\\(4-4i-1)+(3\lambda\mu-3\lambda i+\mu i+1)+5i&=10\\4+3\lambda\mu+i(1-3\lambda+\mu)&=10\end{aligned}$$

Comparing real-coefficients,

$$\begin{aligned}4+3\lambda\mu&=10\\\lambda\mu&=2\ldots(1)\end{aligned}$$

Comparing imaginary-coefficients,

$$\begin{aligned}1-3\lambda+\mu&=0\\\mu&=3\lambda-1\ldots(2)\end{aligned}$$

Substituting (2) into (1),

$$\begin{aligned}\lambda(3\lambda-1)&=2\\3\lambda^2-\lambda-2&=0\\\lambda&=1\quad\text{or}\quad-\frac{2}{3}\\\implies\mu&=2\quad\text{or}\quad-3\end{aligned}$$

Example 3.
Find the square root of $15+8i$.

Solution:

We want to solve for z such that $z=\sqrt{15+8i}$. Let $z=x+yi$.

$$\begin{aligned}z&=\sqrt{15+8i}\\z^2&=15+8i\\(x+yi)^2&=15+8i\\x^2-y^2+2xyi&=15+8i\end{aligned}$$

Comparing real coefficient,

$$x^2-y^2=15\ldots(1)$$

Comparing imaginary coefficient,

$$\begin{aligned}2xy&=8\\y&=\frac{4}{x}\ldots(2)\end{aligned}$$

Substitute (2) into (1),

$$\begin{aligned}x^2-\left(\frac{4}{x}\right)^2&=15\\x^4-15x^2-16&=0\end{aligned}$$

From GC,
$x=4$ or -4.
$\implies y=1$ or -1.

$\therefore z=4+i\quad\text{or}\quad-4-i$.

Example 4 (2010/JJC/Prelim/I/8a)**.**
The complex numbers p and q are such that

$$p = 2 + ia, \ q = b - i,$$

where a and b are real numbers.

Given that $pq = 13 + 13i$, find the possible values of a and b. [4]

[$a = 3$ or 10, $b = 5$ or $\frac{3}{2}$]

Example 5.

Two complex numbers w and z are such that

$$2w + z = 12i \qquad \text{and} \qquad w^* + 2z = \frac{-13+4i}{2-i}.$$

Find w and z, giving each answer in the form $x + iy$.

Solution:

$$\begin{aligned} 2w + z &= 12i \\ z &= 12i - 2w \qquad \text{—(1)} \end{aligned}$$

$$w^* + 2z = \frac{-13+4i}{2-i} \qquad \text{—(2)}$$

Sub (1) into (2):

$$w^* + 2(12i - 2w) = \frac{-13+4i}{2-i}$$

Let $w = x + iy$. Then $w^* = x - iy$.

$$\begin{aligned} (x - iy) + 2[12i - 2(x + iy)] &= \frac{-13+4i}{2-i} \\ &\vdots \qquad \text{(We group the terms with } i \text{ together)} \\ (-6x - 5y + 24) + i(3x - 10y + 48) &= -13 + 4i \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} -6x - 5y + 24 &= -13 \ldots (3) \\ 3x - 10y + 48 &= 4 \ldots (4) \end{aligned}$$

Solving the simultaneous equations (3) and (4), we have $x = 2$ and $y = 5$. Therefore,

$$w = 2 + 5i.$$

From (1),

$$\begin{aligned} z &= 12i - 2w \\ &= 12i - 2(2 + 5i) \\ &= -4 + 2i \end{aligned}$$

Example 6 (RJC/2010/P1/Q1).
By writing $z = x + iy, x, y \in \mathbb{R}$, solve the simultaneous equations

$$z^2 + zw - 2 = 0 \qquad \text{and} \qquad z^* = \frac{w}{1+i}.$$

$[z = 1 - i, w = 2i,\ z = -1 + i, w = -2i]$

Properties of Complex Conjugates

(i) $z + z^* = 2\text{Re}(z)$

(ii) $z - z^* = 2i\text{Im}(z)$

(iii) $(z^*)^* = z$

Bring $*$ into the brackets

(a) $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$

(b) $(z_1 z_2)^* = z_1^* . z_2^*$

(c) $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

(d) $(kz)^* = kz^*$, where $k \in \mathbb{R}$

2 Polar and Exponential Form

2.1 Argand Diagram

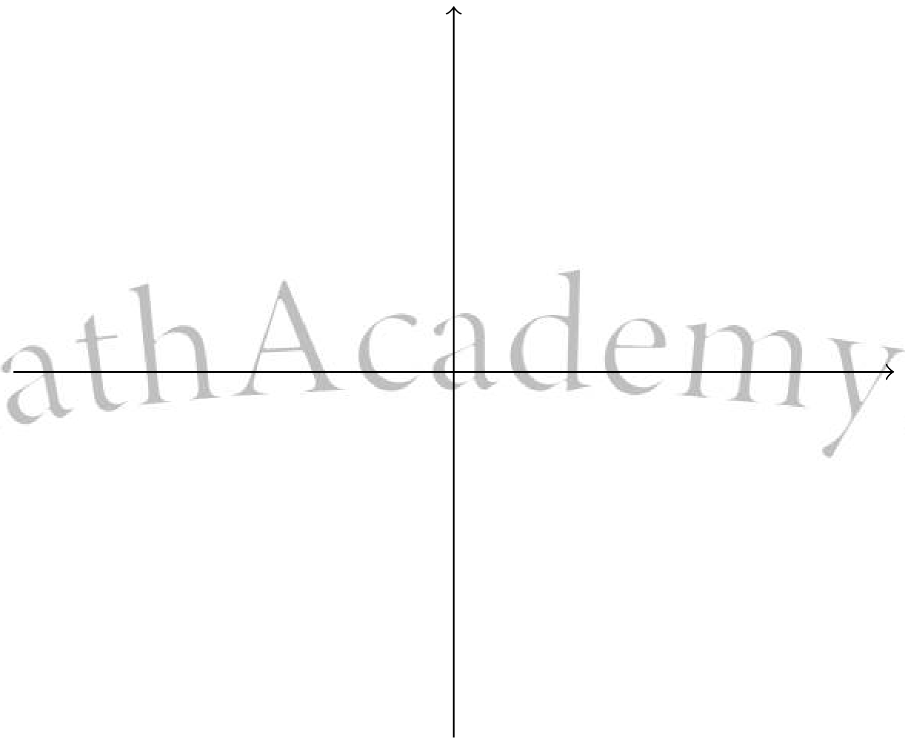
An **Argand diagram** is a cartesian plane that represents complex numbers. The x-axis is the *real axis* and the y-axis is the *imaginary axis*.

A complex number $x + iy$ is represented by the point (x, y). Represent the following complex numbers on the Argand diagram below:

(i) $z_1 = 3 - 4i$

(ii) $z_2 = -2 + 2i$

(iii) $z_2^* = -2 - 2i$

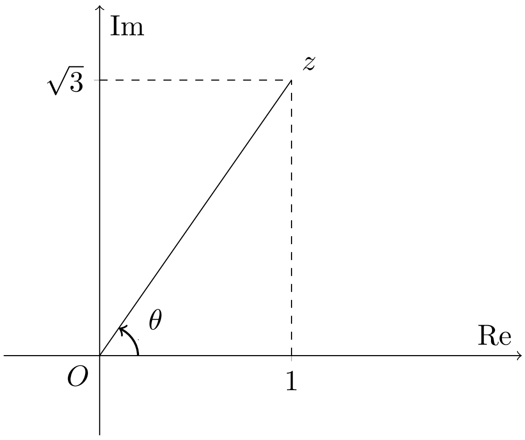


Complex Conjugate

z^*, the complex conjugate of z, is the reflection of z under the x-axis in an Argand diagram.

2.2 Polar Form

Consider the complex number $z = 1 + \sqrt{3}i$.



The length of OZ is given by

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2.$$

The angle θ is given by

$$\tan^{-1} \frac{\sqrt{3}}{1} = 60^\circ.$$

Thus,

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2\cos 60^\circ + i2\sin 60^\circ \\ &= 2(\cos 60^\circ + i\sin 60^\circ) \end{aligned}$$

Polar Form of a Complex Number

If $z = x + iy$, then

$$z = r(\cos\theta + i\sin\theta).$$

$r = |z| = \sqrt{x^2 + y^2}$ is the **modulus** (length) of z.
θ is the **argument** of z, the angle from the positive real axis to OZ.

Since $|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2 = zz^*$, we have,

$$|z|^2 = zz^*.$$

Calculating Argument

Note that θ, $2\pi+\theta$, $4\pi+\theta\ldots$ all describe the same argument. However, only one of these lie in the interval $(-\pi,\pi]$. This is called the principal argument of z.

Principal values of $\arg(z)$

$$-\pi < \arg(z) \leq \pi$$

or $$-180^\circ < \arg(z) \leq 180^\circ$$

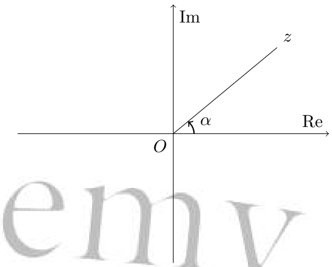
For $z = x+iy$, we first calculate the basic angle

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|.$$

Take mod so that α is always acute. We then determine the quadrant of z.

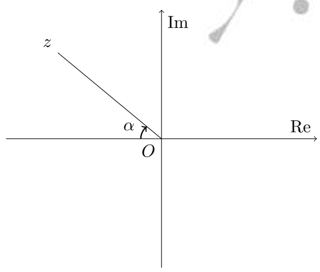
<u>z lies in 1^{st} quadrant</u>

$\arg(z) =$



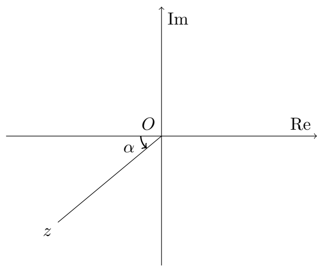
<u>z lies in 2^{nd} quadrant</u>

$\arg(z) =$



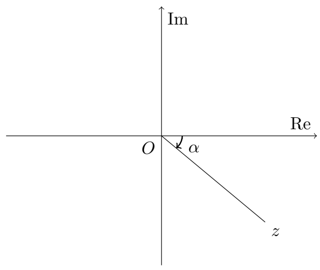
<u>z lies in 3^{rd} quadrant</u>

$\arg(z) =$



<u>z lies in 4^{th} quadrant</u>

$\arg(z) =$



Example 7.
Write $z = -2 - 2i$ and its complex conjugate in polar form.

$$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2} \qquad \arg(z) = -180^\circ + \tan^{-1}\left|\frac{-2}{-2}\right| = -180^\circ + 45^\circ = -135^\circ$$

$z = 2\sqrt{2}\left(\cos(-135^\circ) + i\sin(-135^\circ)\right)$

$z^* = 2\sqrt{2}\left(\cos 135^\circ + i\sin 135^\circ\right)$

Example 8.
Express the following in polar form. Find also their complex conjugate in polar form.

(i) $z_1 = -4$ (ii) $z_2 = 3i$ (iii) $z_3 = -1 + i$ (iv) $z_4 = \sqrt{3} - i$

[(iii) $\sqrt{2}\left(\cos \frac{3\pi}{4} + i\sin \frac{3\pi}{4}\right)$ (iv) $2\left(\cos \frac{-\pi}{6} + i\sin \frac{-\pi}{6}\right)$]

Polar Form of Conjugate

Recall from trigo: $\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta.$

If $z = r(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} z^* &= r[\cos\theta - i\sin\theta] \\ &= r[\cos(-\theta) + i\sin(-\theta)] \end{aligned}$$

Therefore, $|z^*| = |z|$ and $\arg(z^*) = -\arg(z)$.

Question: Is the following polar form? $\sqrt{2}\left(\cos \frac{3\pi}{4} - i\sin \frac{3\pi}{4}\right)$

2.3 Modulus and Argument of Products and Quotients

Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$ be two complex numbers.

Multiplication

$$|z_1 z_2| = |z_1||z_2| \qquad \therefore z_1 z_2 = r_1 r_2[\cos(\theta+\phi) + i\sin(\theta+\phi)]$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Division

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta-\phi) + i\sin(\theta-\phi)]$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Multiple Complex numbers

(a) $|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|$

(b) $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

As a consequence, we have

(c) $|z^n| = |z|^n$

(d) $\arg(z^n) = n\arg(z)$

Remark: "arg" works the same way as the "log" function.

Example 9.
Find the modulus and argument of $z = (-2+2i)(-1-i\sqrt{3})^3$. Hence find z in polar form.

$$\begin{aligned}
\left|(-2+2i)(-1-i\sqrt{3})^3\right| &= \left|-2+2i\right|\left|(-1-i\sqrt{3})^3\right| \\
&= \left|-2+2i\right|\left|-1-i\sqrt{3}\right|^3 \\
&= \sqrt{2^2+2^2} \times \left(\sqrt{1^2+\sqrt{3}^2}\right)^3 \\
&= 16\sqrt{2} \\
\arg\left((-2+2i)(-1-i\sqrt{3})^3\right) &= \arg(-2+2i) + \arg(-1-i\sqrt{3})^3 \\
&= \arg(-2+2i) + 3\arg(-1-i\sqrt{3}) \\
&= (\pi - \tan^{-1}1) + 3(-\pi + \tan^{-1}\sqrt{3}) \\
&= -\frac{5}{4}\pi
\end{aligned}$$

For $-\pi < \arg(z) \le \pi$,

$$\arg(z) = -\frac{5}{4}\pi + 2\pi = \frac{3}{4}\pi.$$

$\therefore (-2+2i)(-1-i\sqrt{3})^3 = 16\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

Example 10.
Find the modulus and argument of the following complex numbers. Hence, express them in (a) polar form, (b) the form $a + bi$.

(i) $(2+2\sqrt{3}i)(-\sqrt{3}+i)^4$

(ii) $\frac{2i}{1+i}$

[i) $64\left(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)$;$32-32\sqrt{3}i$ ii) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$; $1+i$]

Example 11 (2014/SRJC/Prelim/II/4a)**.**
The complex number z is such that $|z^2| = 3$ and $\arg(-iz) = \frac{\pi}{4}$.

Find w in the form $a + bi$, where $a, b \in \mathbb{R}$, if $|wz| = 2\sqrt{3}$ and $\arg\left(\frac{z^2}{w}\right) = \frac{5}{6}\pi$.

Solution:

$$\begin{aligned} |z^2| &= 3 \\ |z|^2 &= 3 \\ |z| &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \arg(-iz) &= \frac{\pi}{4} \\ \arg(-i) + \arg z &= \frac{\pi}{4} \\ -\frac{\pi}{2} + \arg z &= \frac{\pi}{4} \\ \arg z &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \end{aligned}$$

$$\begin{aligned} |wz| &= 2\sqrt{3} \\ |w||z| &= 2\sqrt{3} \\ |w|(\sqrt{3}) &= 2\sqrt{3} \\ |w| &= 2 \end{aligned}$$

$$\begin{aligned} \arg\left(\frac{z^2}{w}\right) &= \frac{5}{6}\pi \\ \arg(z^2) - \arg w &= \frac{5}{6}\pi \\ 2\arg z - \arg w &= \frac{5}{6}\pi \\ 2\left(\frac{3\pi}{4}\right) - \arg w &= \frac{5}{6}\pi \\ \arg w &= \frac{3\pi}{2} - \frac{5}{6}\pi = \frac{2}{3}\pi \end{aligned}$$

$\therefore w = 2\left[\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right] = -1 + \sqrt{3}i.$

Example 12 (2013/TJC/Prelim/I/5).
A graphic calculator is **not** to be used in answering this question.

Two complex numbers p and q are given by $p = 1 - i$ and $q = -1 + \sqrt{3}i$, and $z = \frac{p}{q}$.

(i) Express z in the form $x + yi$, where x and y are exact real values to be determined.

(ii) By considering the moduli and arguments of p and q, find the exact values of $|z|$ and $\arg z$, where $-\pi < \arg z \leq \pi$.

(iii) Hence, show that $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$.

[(i) $\frac{-1-\sqrt{3}}{4} + \frac{1-\sqrt{3}}{4}i$ (ii) $|z| = \frac{\sqrt{2}}{2}$; $\arg z = -\frac{11}{12}\pi$]

2.4 Exponential Form

Euler's formula

(Proof is not required in syllabus)

$$\cos\theta + i\sin\theta = e^{i\theta}$$

Hence, we can convert from polar form to exponential form:

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where r is the modulus and θ is the argument of the complex number.

Remark:

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Since $|e^{i\theta}| = 1$, the co-efficient beside the exponential will always give you the modulus of the complex number.

For example, $|6\sqrt{2}e^{i\frac{\pi}{4}}| = 6\sqrt{2}$.

Example 13.
Express the following in exponential form. Find also their complex conjugate in exponential form.

(i) $z_1 = -1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) $z_2 = \sqrt{3} - i = 2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$

Write down the modulus and argument of the following complex numbers:

(i) $z_3 = 4e^{i\frac{2\pi}{3}}$

(ii) $z_4 = -2e^{-i\pi}$

Example 14.
A complex number z is given by $z = \frac{2}{1+e^{4\alpha i}}$, where $0 < \alpha < \pi$.

(i) Show that $e^{2\alpha i} + e^{-2\alpha i}$ is a real number for all α.

(ii) Hence, show that $\text{Re}(z) = 1$.

Solution:

(i)

$$\begin{aligned} e^{2\alpha i} + e^{-2\alpha i} &= (\cos 2\alpha + i \sin 2\alpha) + [\cos(-2\alpha) + i \sin(-2\alpha)] \\ &= (\cos 2\alpha + i \sin 2\alpha) + [\cos 2\alpha - i \sin 2\alpha] \\ &= 2 \cos 2\alpha \end{aligned}$$

(ii)

$$\begin{aligned} z &= \frac{2}{1 + e^{4\alpha i}} \\ &= \frac{2}{e^{2\alpha i}(e^{-2\alpha i} + e^{2\alpha i})} \\ &= \frac{2}{e^{2\alpha i}(2 \cos 2\alpha)} \\ &= \frac{e^{-2\alpha i}}{\cos 2\alpha} \\ &= \frac{\cos(-2\alpha) + i \sin(-2\alpha)}{\cos 2\alpha} \\ &= \frac{\cos(2\alpha) - i \sin(2\alpha)}{\cos 2\alpha} \\ &= 1 + i \left(\frac{-\sin 2\alpha}{\cos 2\alpha} \right) \end{aligned}$$

$\therefore$ Re(z)=1.

Example 15 (2012/IJC/Prelim/I/7c).
Show that $3 + 3e^{i\theta} = 6e^{i\frac{\theta}{2}} \cos \frac{\theta}{2}$, where $0 < \theta < \pi$.

Example 16 (2014/MJC/Prelim/I/8).
Show that $e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} = 2i \sin \frac{\theta}{2}$. Hence, show that $\frac{e^{i\theta}}{1-e^{i\theta}} = \frac{1}{2}\left(i \cot \frac{\theta}{2} - 1\right)$.

2.5 Purely Real/Imaginary

Let $z = re^{i\theta}$. For what values of n, is

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n[\cos n\theta + i \sin n\theta],$$

purely real or imaginary?

Purely real:

$$\begin{aligned} \sin n\theta &= 0 \\ n\theta &= k\pi, \quad k \in \mathbb{Z}. \end{aligned}$$

Purely imaginary:

$$\begin{aligned} \cos n\theta &= 0 \\ n\theta &= \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}. \end{aligned}$$

In both cases, we then proceed to make n the subject.

Example 17.
It is given that $z = 1 + i$.

i. Find the set of values of n such that z^n is real.
ii. Hence, find the least positive integer value of n such that z^n is real.
iii. Hence, find the least positive integer value of n such that z^n is real and positive.

Solution:

i. Converting z into its exponential form,

$$\begin{aligned} z &= \sqrt{2}e^{\frac{\pi}{4}i} \\ \implies z^n &= (\sqrt{2})^n e^{\frac{n\pi}{4}i} \\ &= (\sqrt{2})^n \left(\cos \frac{\pi}{4}n + i \sin \frac{\pi}{4}n\right) \end{aligned}$$

For z^n to be purely real,

$$\begin{aligned} \sin \frac{\pi}{4}n &= 0 \\ \frac{\pi}{4}n &= k\pi, \\ n &= 4k, \quad k \in \mathbb{Z} \end{aligned}$$

The set of values is $\{n : n = 4k, k \in \mathbb{Z}\}$.

ii. Smallest integer n is 4.

iii.

Example 18 (2011/MI/Prelim/I/12b).
It is given that $z = -\sqrt{3} + i$.

Find the least positive integer value of n such that z^n is purely imaginary. [3]

[$n = 3$]

3 Finding roots of polynomial equations

Consider the function $f(z) = z^3 - 3z^2 + z + 5$. We check that $w = 2 + i$ and $w^* = 2 - i$ are roots of $f(z) = 0$ since:

$$(2+i)^3 - 3(2+i)^2 + (2+i) + 5 = \cdots = 0$$

$$(2-i)^3 - 3(2-i)^2 + (2-i) + 5 = \cdots = 0$$

Is it always true that if w is a root of $f(z) = 0$, then w^* is also a root of the equation?

Important Theorems

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

(i) **Fundamental Theorem of Algebra:**

$f(z) = 0$ has n roots.

(ii) **Conjugate Root Theorem:**

If $a_0, a_1, \ldots, a_n \in \mathbb{R}$, then any complex roots of $f(z) = 0$ exists in conjugate pairs.

For example, a **cubic** equation with **real** coefficients fulfills one of the following:

(i) 3 real roots or,

(ii) 1 real root and 2 complex conjugate roots.

Example 19.
Explain why the equation $\left(\frac{3w+2}{w}\right)^4 - 81 = 0$ has only three roots.

Solution:

$$\left(\frac{3w+2}{w}\right)^4 - 81 = 0$$
$$(3w+2)^4 = 81w^4$$

Note that the term with highest degree, w^4, has the same coefficient on the left and right hand side. They cancel each other other and the largest degree of the equation is 3. By fundamental theorem of algebra, the equation only has three roots.

Example 20 (N2008/P1/Q8 modified)**.**
Given that $1+\sqrt{3}i$ is a root of the equation

$$2z^3 + az^2 + bz + 4 = 0,$$

find the values of the real numbers a and b. Hence, without using a calculator, find the other roots.

Solution:

$$\begin{aligned} 2(1+\sqrt{3}i)^3 + a(1+\sqrt{3}i)^2 + b(1+\sqrt{3}i) + 4 &= 0 \\ \text{Using GC,} \qquad -16 + a(-2+2\sqrt{3}i) + b(1+\sqrt{3}i) + 4 &= 0 \\ (-12-2a+b) + i(2\sqrt{3}a + b\sqrt{3}) &= 0 \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} -12-2a+b &= 0 \\ 2\sqrt{3}a + b\sqrt{3} &= 0 \end{aligned}$$

Solving simultaneous equations, we get $a=-3$ and $b=6$.

Since the coefficients of the equation are real and $(1+\sqrt{3}i)$ is a root, then $(1-\sqrt{3}i)$ is also a root. So,

$$\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right) = (z-1)^2-(-3) = z^2-2z+4$$

is a factor of the equation.
By inspection,

$$2z^3-3z^2+6z+4 = (z^2-2z+4)(2z+1).$$

Hence, the last factor is $2z+1$. (We compare coefficient of z^3 and the constant.)
$\therefore$ the roots are $(1+\sqrt{3}i), (1-\sqrt{3}i)$ and $-\frac{1}{2}$.

If the question did not restrict the use of GC, how should you find all other roots?

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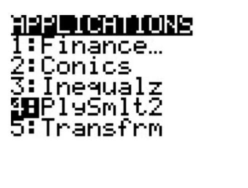
<td>
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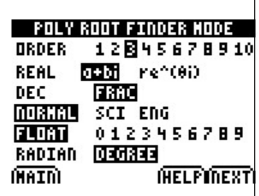
<td>
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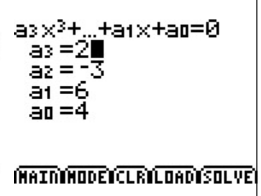
<td>
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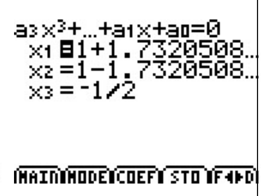
<td>
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Example 21 (Quadratic formula holds for complex numbers)**.**
Solve the quadratic equation $z^2 + 2z + 2 = 0$.

Example 22 (2011/MI/Prelim/I/12a)**.**
It is given that $1 - i$ is a roots to the equation $2z^4 - 5z^3 + 8z^2 + kz + 4 = 0$ where k is a real constant.

i. Show that $k = -6$. [1]

ii. Without using a calculator, find the exact values of the other roots of the equation. [5]

[(ii) $1 + i$ and $\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$]

Example 23 (2012/TJC/Prelim/I/2).
Consider the equation

$$2z^3 + (1-2i)z^2 - (a+bi)z + 2 + 2i = 0,$$

where a and b are real.

Given that -2 is a root of the equation, find the values of a and b.

Given also that $1+i$ is another root, find the third root of the equation.

Solution:

Since -2 is a root to the equation,

$2(-2)^3 + (1-2i)(-2)^2 - (a+bi)(-2) + 2 + 2i = 0$

$\implies -16 + 4 - 8i + 2a + 2bi + 2 + 2i = 0$

$\implies -10 + 2a + i(-6+2b) = 0$

By comparing real and imaginary parts,

$-10 + 2a = 0 \quad \implies a = -5$

$-6 + 2b = 0 \quad \implies b = 3$

Since -2 and $1+i$ are both roots to the equation, $z+2$ and $z-1-i$ are factors of the equation.

Let the last factor be $az+b$, where $a, b \in \mathbb{C}$. Then,

$2z^3 + (1-2i)z^2 - (-5+3i)z + 2 + 2i = 0 = (z+2)(z-1-i)(az+b)$

By inspection, $a = 2$.

By comparing coefficient of the constant term,

$$\begin{aligned} 2+2i &= (2)(-1-i)(b) \\ 2+2i &= (-2-2i)b \\ b &= \frac{2+2i}{-2-2i} = -1 \end{aligned}$$

$\therefore 2z - 1 = 0 \implies z = \frac{1}{2}$

Example 24.
From example 20, the equation

$$2z^3 - 3z^2 + 6z + 4 = 0$$

has roots $(1+\sqrt{3}i), (1-\sqrt{3}i)$ and $-\frac{1}{2}$. Using this result, hence, deduce the roots of the following equations:

i) $2(w+1)^3 - 3(w+1)^2 + 6(w+1) + 4 = 0$

ii) $-2iw^3 + 3w^2 + 6iw + 4 = 0$

iii) $4w^3 + 6w^2 - 3w + 2 = 0$

Example 25.
From example 23, the equation

$$2z^3 + (1-2i)z^2 - (-5+3i)z + 2 + 2i = 0,$$

has roots $-2, 1+i$ and $\frac{1}{2}$. Using this result, hence, deduce the roots of the following:

$$2z^3 + (1+2i)z^2 - (-5-3i)z + 2 - 2i = 0.$$

Example 26 (2014/RI/Prelim/8).
One root of the equation $z^2 + az^* + b = 0$, where a and b are real, is w.

Show that w^* is also a root of this equation.

Solution:

Since w is a root of the equation $z^2 + az^* + b = 0$, w satisfies the equation.

$$\begin{aligned} w^2 + aw^* + b &= 0 \\ \left(w^2 + aw^* + b\right)^* &= 0^* \\ (w^2)^* + (aw^*)^* + b^* &= 0 \\ (w^*)^2 + a(w^*)^* + b &= 0 \quad \text{since } a \text{ and } b \text{ are real, } a^* = a,\ b^* = b. \end{aligned}$$

Hence, w^* is a root of the equation $z^2 + az^* + b = 0$.

