Complex Numbers Lesson 1

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Basic operations

Definition

Let $i = \sqrt{-1}$. A complex number is a number of the form

$$z = x + iy, \quad \text{where } x, y \in \mathbb{R}$$

x is the real part of z, denoted by $\text{Re}(z)$.
y is the imaginary part of z, denoted by $\text{Im}(z)$.

Powers of i

i^0	1
i	i
i^2	-1
i^3	$-i$

i^4	1
i^5	i
i^6	-1
i^7	$-i$

The pattern $1, i, -1, -i, \ldots$ will repeat itself.

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Complex conjugate z^*

If $z = x + iy$, then its *complex conjugate* z^* is defined as

$$z^* = x - iy$$

Observe the following:

$$zz^* = x^2 + y^2$$

Thus, the product of a complex number and its complex conjugate is a real number.

Proof:

$$\begin{aligned} zz^* &= (x + yi)(x - yi) \\ &= x^2 - (yi)^2 \\ &= x^2 - (-y^2) \\ &= x^2 + y^2 \end{aligned}$$

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Operations of Complex Numbers

(i) **Equality**

$$a + bi = c + di \iff a = c \quad \text{AND} \quad b = d$$

(ii) **Addition/Subtraction**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

(iii) **Multiplication**

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

(iv) **Division** (Multiply denominator by its complex conjugate)

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

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Example (1)

Express the following in the form $x + iy$, where $x, y \in \mathbb{R}$.

a) $z = (3 - 3i) + (-3 + i)$

c) $z = (2 + 2i)(1 - 3i)$

e) $z = \frac{2+10i}{5-3i}$

Example (1)

Express the following in the form $x + iy$, where $x, y \in \mathbb{R}$.

b) $z = (2 - \sqrt{2}i) - (1 - 3i)$

d) $z = (\sqrt{3} + 2i)(\sqrt{3} - 2i)$

f) $z = \frac{3+2i}{4-i}$

Example (2)

Given that $(2-i)^2+(3\lambda+i)(\mu-i)+5i=10$, find the exact values of λ and μ.

$$(2-i)^2+(3\lambda+i)(\mu-i)+5i=10$$
$$(4-4i-1)+(3\lambda\mu-3\lambda i+\mu i+1)+5i=10$$
$$4+3\lambda\mu+i(1-3\lambda+\mu)=10$$

Comparing real-coefficients,

$$4+3\lambda\mu=10$$
$$\lambda\mu=2\ldots(1)$$

Comparing imaginary-coefficients,

$$1-3\lambda+\mu=0$$
$$\mu=3\lambda-1\ldots(2)$$

Substituting (2) into (1),

$$\lambda(3\lambda-1)=2$$
$$3\lambda^2-\lambda-2=0$$
$$\lambda=1 \quad \text{or} \quad -\frac{2}{3}$$
$$\Longrightarrow \mu=2 \quad \text{or} \quad -3$$

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Comparing real-coefficients,

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Example (3)

Find the square root of $15 + 8i$.

We want to solve for z such that $z = \sqrt{15 + 8i}$. Let $z = x + yi$.

$$z = \sqrt{15 + 8i}$$
$$z^2 = 15 + 8i$$
$$(x + yi)^2 = 15 + 8i$$
$$x^2 - y^2 + 2xyi = 15 + 8i$$

Comparing real coefficient,

$$x^2 - y^2 = 15 \ldots (1)$$

Comparing imaginary coefficient,

$$2xy = 8$$
$$y = \frac{4}{x} \ldots (2)$$

Substitute (2) into (1),

$$x^2 - \left(\frac{4}{x}\right)^2 = 15$$
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From GC,

$x = 4$ or -4.

$\implies y = 1$ or -1.

$\therefore z = 4 + i \quad$ or $\quad -4 - i$.

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Example (4)

The complex numbers p and q are such that

$$p = 2 + ia, \ \ q = b - i,$$

where a and b are real numbers.
Given that $pq = 13 + 13i$, find the possible values of a and b. [4]

[$a = 3$ or 10, $b = 5$ or $\frac{3}{2}$]

$$\begin{aligned} pq &= (2 + ia)(b - i) \\ &= 2b - 2i + abi + a \\ &= 2b + a + i(ab - 2) \end{aligned}$$

$13 + 13i = 2b + a + i(ab - 2)$

Comparing real and imaginary parts,

$2b + a = 13$

$a = 13 - 2b - - - - - - (1)$

$ab - 2 = 13$

$ab = 15 - - - - - - (2)$

Sub (1) into (2)

$(13 - 2b)b = 15$

$-2b^2 + 13b - 15 = 0$

$b = 5$ or $\frac{3}{2}$

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Example (5)

Two complex numbers w and z are such that

$$2w + z = 12i \qquad \text{and} \qquad w^* + 2z = \frac{-13+4i}{2-i}.$$

Find w and z, giving each answer in the form $x + iy$.

$$2w + z = 12i$$

$$z = 12i - 2w \qquad \text{—(1)}$$

$$w^* + 2z = \frac{-13+4i}{2-i} \qquad \text{—(2)}$$

Sub (1) into (2):

$$w^* + 2(12i - 2w) = \frac{-13+4i}{2-i}$$

Let $w = x + iy$. Then $w^* = x - iy$.

$$(x - iy) + 2[12i - 2(x + iy)] = \frac{-13+4i}{2-i}$$

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$$(x - iy) + 2[12i - 2(x + iy)] = \frac{-13+4i}{2-i}$$

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Two complex numbers w and z are such that

$$2w + z = 12i \qquad \text{and} \qquad w^* + 2z = \frac{-13 + 4i}{2 - i}.$$

Find w and z, giving each answer in the form $x + iy$.

$$2w + z = 12i$$

$$z = 12i - 2w \qquad \text{—(1)}$$

$$w^* + 2z = \frac{-13 + 4i}{2 - i} \qquad \text{—(2)}$$

Sub (1) into (2):

$$w^* + 2(12i - 2w) = \frac{-13 + 4i}{2 - i}$$

Let $w = x + iy$. Then $w^* = x - iy$.

$$(x - iy) + 2[12i - 2(x + iy)] = \frac{-13 + 4i}{2 - i}$$

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$$\vdots \quad \text{(We group the terms with } i \text{ together)}$$

$$(-6x - 5y + 24) + i(3x - 10y + 48) = -13 + 4i$$

Comparing coefficients,

$$-6x - 5y + 24 = -13 \ldots (3)$$

$$3x - 10y + 48 = 4 \ldots (4)$$

Solving the simultaneous equations (3) and (4), we have $x = 2$ and $y = 5$. Therefore,

$$w = 2 + 5i.$$

From (1),

$$\begin{aligned} z &= 12i - 2w \\ &= 12i - 2(2 + 5i) \\ &= -4 + 2i \end{aligned}$$

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Example (6)

By writing $z = x + iy, x, y \in \mathbb{R}$, solve the simultaneous equations

$$z^2 + zw - 2 = 0 \qquad \text{and} \qquad z^* = \frac{w}{1+i}.$$

$[z = 1 - i, w = 2i,\ z = -1 + i, w = -2i]$

From second equation, $w = z^*(1+i)$
Substitute into first equation,

$$z^2 + z[z^*(1+i)] - 2 = 0$$

Let $z = x + iy$.

$$(x+iy)^2 + (x^2+y^2)(1+i) - 2 = 0$$

$$x^2 + 2xyi - y^2 + x^2 + y^2 + (x^2+y^2)i - 2 = 0$$

$$2x^2 - 2 + i(2xy + x^2 + y^2) = 0$$

Comparing coefficient of real and imaginary parts,

$$2x^2 - 2 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$2xy + x^2 + y^2 = 0$$

$$(x+y)^2 = 0$$

if $x = 1, y = -1$
if $x = -1, y = 1$

When $z = 1 - i$,

$$w = z^*(1+i)$$

$$= (1+i)(1+i)$$

$$= 1 + 2i - 1$$

$$= 2i$$

When $z = -1 + i$,

$$w = z^*(1+i)$$

$$= (-1-i)(1+i)$$

$$= -2i$$

Properties of Complex Conjugates

(i) $z + z^* = 2\text{Re}(z)$

(ii) $z - z^* = 2i\text{Im}(z)$

(iii) $(z^*)^* = z$

Bring $*$ into the brackets

(a) $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$

(b) $(z_1 z_2)^* = z_1^* . z_2^*$

(c) $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

(d) $(kz)^* = kz^*, \quad \text{where } k \in \mathbb{R}$

Example

Prove (i), (ii) and (a) in the above properties.

$z + z^* = 2\text{Re}(z)$

Let $z = x + yi$. Then $z^* = x - yi$.

$z - z^* = 2i\text{Im}(z)$

Let $z = x + yi$. Then $z^* = x - yi$.

$(z_1 \pm z_2)^* = z_1^* \pm z_2^*$