INEQUALITIES

1 Using a number line

Step 1. Find critical points. (Values of x that makes either numerator or denominator 0)

Step 2. Choose a region, check the sign.

Step 3. Alternate the signs, except at even powers.

Step 4. **Include** points that are $= 0$, and **reject** points that makes denominator 0.

Example 1.
Solve the inequality $\frac{(2+x)^2(x+1)(2-x)}{(3-x)} \leq 0$.

Example 2.
Solve the following inequalities

(i) $\frac{(x-2)^2(x-5)}{x-1} \geq 0$
[$x < 1$ or $x = 2$ or ≥ 5]

(ii) $\frac{-(x+1)^2(x-5)^3}{x} < 0$
[$x < -1$ or $-1 < x < 0$ or $x > 5$]

2 Inequalities Involving Rational Functions (Fractions)

Steps in solving fractional inequalities

Example: Solve the inequality $\frac{x^2-3}{x+2} \geq \frac{2x}{x+2}$.

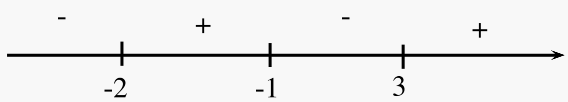
Step 1. Write down the values of x that makes the denominator zero.

$$x \neq -2$$

Step 2. Combine the terms into one fraction

$$\begin{aligned} \frac{x^2-3}{x+2} &\geq \frac{2x}{x+2} \\ \frac{x^2-3}{x+2} - \frac{2x}{x+2} &\geq 0 \\ \frac{x^2-2x-3}{x+2} &\geq 0 \\ \frac{(x-3)(x+1)}{x+2} &\geq 0 \end{aligned}$$

Step 3. Draw the number line



Step 4. Identify the region, ignoring the values in Step 1.

$$-2 \leq x \leq -1 \quad \text{or} \quad x \geq 3$$

In step 1, $x \neq -2$. Therefore,

$$-2 < x \leq -1 \quad \text{or} \quad x \geq 3$$

Remark: What happens if you obtain the following equation:

$$\frac{x^2-4x+1}{x+1} > 0.$$

Example 3 (2011/ACJC/Prelim/I/1).
Without using the graphic calculator, find the range of values of x which satisfy the inequality

$$\frac{1}{x-1} \geq \frac{2}{x+2}.$$

[$x < -2$ or $1 < x \leq 4$]

Common Mistake

Be very careful when factorising a **negative** x^2 equation.

Example 4 (2009/NYJC/I/2).
By using algebraic method, solve the inequality $\frac{-x^2+5}{x} \leq -2$. [$1-\sqrt{6} \leq x < 0$ or $x \geq 1+\sqrt{6}$]

Common Replacements

$$ax+b, \quad \ln x, \quad e^x, \quad e^{-x}, \quad \frac{1}{x}, \quad \frac{1}{x^2}, \quad x^2, \quad |x|$$

You should also consider the negative of all of the above.

Example 5 (HENCE questions)**.**
Solve the inequality $\frac{(x+1)(x-3)}{(x-2)^2} < 0$. [$-1 < x < 3$ and $x \neq 2$]
Hence solve, exactly,

(a) $\frac{(\ln x+1)(\ln x-3)}{(\ln x-2)^2} < 0$,

(b) $\frac{(x^2+1)(x^2-3)}{(x^2-2)^2} < 0$,

(c) $\frac{(x-1)(x+3)}{(x+2)^2} < 0$,

(d) $\frac{(x+1)(1-3x)}{(1-2x)^2} < 0$.

3 Simplifying Inequalities

Simplifying Inequalities

When dividing both sides of the equation by a **POSITIVE** term, the sign remains unchanged.

Example:
Since $(x-1)^2+1>0$ for all x,

$$(x+1)[(x-1)^2+1]>0 \quad \Longrightarrow \quad (x+1)>0$$
$$(x+1)[(x-1)^2+1]\leq 0 \quad \Longrightarrow \quad (x+1)\leq 0$$

Danger: This does not work for terms such as $(x-2)^2$. For example:

$$(x+1)(x-2)^2>0 \quad \not\Rightarrow \quad (x+1)>0$$

The term that you want to remove must be **POSITIVE**, it cannot be zero for some values of x.

Example 6.
Prove that x^2+4x+5 is always positive for all real x and hence find the set of values of x for which

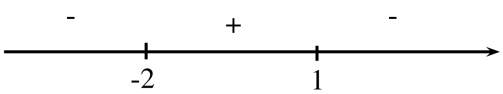
$$(x^2+4x+5)(x^2+x-2)<0.$$

Solution:
Use complete the square method to prove that a quadratic equation is always positive.

$$\begin{aligned} x^2+4x+5 &= x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+5 \\ &= (x+2)^2-2^2+5 \\ &= (x+2)^2+1 \end{aligned}$$

Therefore,

$$\begin{aligned} (x^2+4x+5)(x^2+x-2)&<0 \\ [(x+2)^2+1](x^2+x-2)&<0 \\ (x^2+x-2)&<0 \\ (x+2)(x-1)&<0 \end{aligned}$$



$$\{x\in\mathbb{R} \mid -2<x<1\}$$

The notation for a *set* takes the following format:

$$\{\text{name of variable} \quad | \quad \text{condition of variable}\}$$

Example 7 (2015/SAJC/Prelim/I/1).
Without using a calculator, solve the inequality

$$\frac{5x^2 - x - 14}{2x^2 + x - 3} \leq 3.$$

$[x < -\frac{3}{2} \text{ or } x > 1]$

Example 8.

[2014/MI/Prelim/P2/Q1modifed]

(i) Without the use of a calculator, solve the inequality

$$\frac{-x^2+4x+2}{x+4} \geq 1$$

(ii) Hence solve the inequality

$$\frac{-2\cos^2\theta+4\cos\theta+1}{\cos\theta+2} \geq 1$$

where $0 \leq \theta \leq 2\pi$.

Solution: **(We only focus on the Hence part)**

Answer to part (i): $x < -4$ or $1 \leq x \leq 2$.

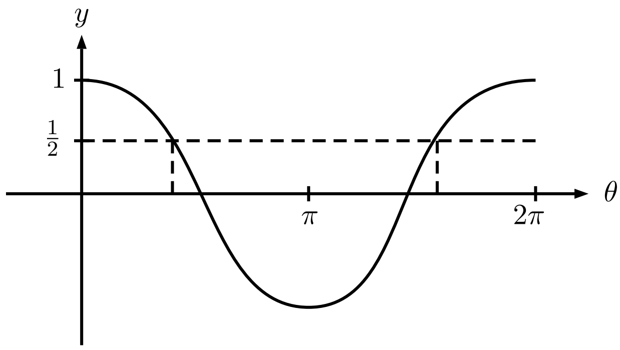
Replace x by $2\cos\theta$ in part (i),

$$\frac{-(2\cos\theta)^2+4(2\cos\theta)+2}{(2\cos\theta)+4} \geq 1$$
$$\frac{-4\cos^2\theta+8\cos\theta+2}{2\cos\theta+4} \geq 1$$
$$\frac{-2\cos^2\theta+4\cos\theta+1}{\cos\theta+2} \geq 1$$

Replacing x by $2\cos\theta$ in part (i),

$2\cos\theta < -4$ or $1 \leq 2\cos\theta \leq 2$

$\cos\theta < -2$ (No solutions) $\quad \frac{1}{2} \leq \cos\theta \leq 1$



4 Solving Modulus Algebraically

4.1 Open up Modulus

If a is a positive constant, then

a) $|x| < a \quad \Rightarrow \quad -a < x < a$

b) $|x| > a \quad \Rightarrow \quad x < -a \quad \text{or} \quad x > a$

c) $|x - b| < a \quad \Rightarrow \quad b - a < x < b + a$

d) $|x - b| > a \quad \Rightarrow \quad x < b - a \quad \text{or} \quad x > b + a$

In general,

e) $|f(x)| < g(x) \quad \Rightarrow \quad -g(x) < f(x) < g(x)$

f) $|f(x)| > g(x) \quad \Rightarrow \quad f(x) < -g(x) \quad \text{or} \quad f(x) > g(x)$

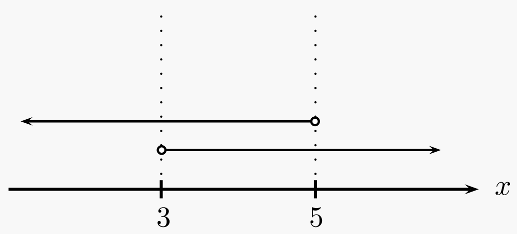
4.2 When to use AND versus OR?

'And' Versus 'Or' Inequalities

i) Take the **INTERSECTION** of both solutions.

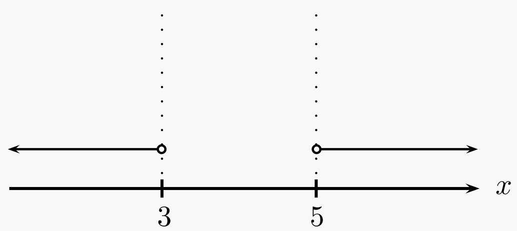
$$3 < x < 5$$

$$3 < x \quad \text{and} \quad x < 5$$



ii) Take the **UNION** of both solutions.

$$x < 3 \quad \text{or} \quad x > 5$$

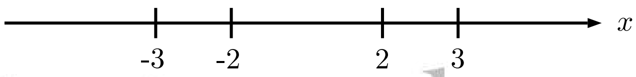


Example 9.
Solve $2 \leq |x| < 3$.

Solution:

$$2 \leq |x| \qquad \text{And} \qquad |x| < 3$$

$$x \leq -2 \quad \text{or} \quad x \geq 2 \qquad\qquad -3 < x < 3$$



Example 10.
Given that a is a positive constant, solve the inequality $|2x + 2a| < -x + a$.

$[-3a < x < -\frac{a}{3}]$

Example 11.
Solve the inequality $-x+2 \leq -\frac{5}{x}$ exactly.

Hence deduce the solution of $-|x|+2 < -\frac{5}{|x|}$.

Solution:

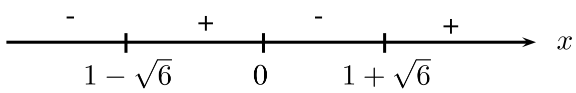
$$x \neq 0$$

$$\begin{aligned}
-x+2 &\leq \frac{-5}{x} \\
\frac{x(-x+2)}{x} + \frac{5}{x} &\leq 0 \\
\frac{-x^2+2x+5}{x} &\leq 0
\end{aligned}$$

Let $-x^2+2x+5=0$

$$\begin{aligned}
x &= \frac{-2 \pm \sqrt{2^2-4(-1)(5)}}{2(-1)} \\
&= \frac{-2 \pm 2\sqrt{6}}{-2} \\
&= 1 \pm \sqrt{6}
\end{aligned}$$

$$\begin{aligned}
\frac{-(x-1-\sqrt{6})(x-1+\sqrt{6})}{x} &\leq 0 \\
\frac{(x-1-\sqrt{6})(x-1+\sqrt{6})}{x} &\geq 0
\end{aligned}$$



$\therefore 1-\sqrt{6} \leq x < 0$ or $\quad x \geq 1+\sqrt{6}$

From $-x+2 \leq \frac{-5}{x}$, replace x by $|x|$ to obtain $-|x|+2 \leq \frac{-5}{|x|}$.

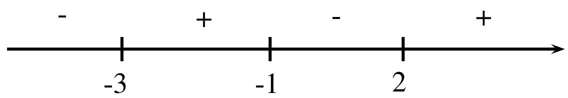
$1-\sqrt{6} \leq |x| < 0$ (N.A.) or $|x| \geq 1+\sqrt{6}$

$x \leq -1-\sqrt{6}$ or $x \geq 1+\sqrt{6}$

Example 12.
(i) Solve the inequality $\frac{6}{x+1} > x$. (ii) Hence solve $\frac{6}{|x|+1} > |x|$.

Solution:
(i) $x \neq -1$.

$$\begin{aligned}\frac{6}{x+1} - x &> 0\\ \frac{6}{x+1} - \frac{x(x+1)}{x+1} &> 0\\ \frac{6-x^2-x}{x+1} &> 0\\ \frac{-(x-2)(x+3)}{x+1} &> 0\\ \frac{(x-2)(x+3)}{x+1} &< 0\end{aligned}$$



$$x < -3 \quad \text{or} \quad -1 < x < 2.$$

(ii)
Replace x by $|x|$ in $\frac{6}{x+1} > x$ to obtain $\frac{6}{|x|+1} > |x|$, therefore,

$$|x| < -3 \qquad \text{or} \qquad -1 < |x| < 2$$

Squaring both sides

When both sides of the inequality only have modulus, we square both sides.

$$\begin{aligned} |x| < |y| &\implies x^2 < y^2 \\ &\implies x^2 - y^2 < 0 \\ &\implies (x-y)(x+y) < 0 \end{aligned}$$

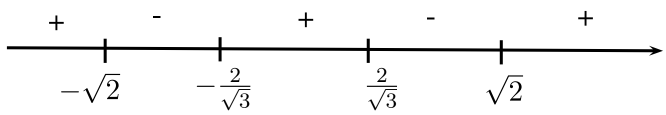
Note: We can square both sides, because they are both positive.

Example 13.
Solve, without using a calculator, the inequality $\frac{|2x^2-3|}{|x^2-1|} < 1$.

Solution:

$$x \neq 1, -1$$

$$\begin{aligned} |2x^2-3| &< |x^2-1| \\ \left(2x^2-3\right)^2 &< \left(x^2-1\right)^2 \\ \left(2x^2-3\right)^2 - \left(x^2-1\right)^2 &< 0 \\ \left(2x^2-3-x^2+1\right)\left(2x^2-3+x^2-1\right) &< 0 \\ \left(x^2-2\right)\left(3x^2-4\right) &< 0 \\ \left(x^2-2\right)\left(x^2-\frac{4}{3}\right) &< 0 \\ \left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x-\frac{2}{\sqrt{3}}\right)\left(x+\frac{2}{\sqrt{3}}\right) &< 0 \end{aligned}$$



$$\therefore -\sqrt{2} < x < -\frac{2}{\sqrt{3}} \quad \text{or} \quad \frac{2}{\sqrt{3}} < x < \sqrt{2}, \quad x \neq 1, -1$$

5 Inequalities Involving GC

Question: How do you know if you should use a GC to solve inequalities?

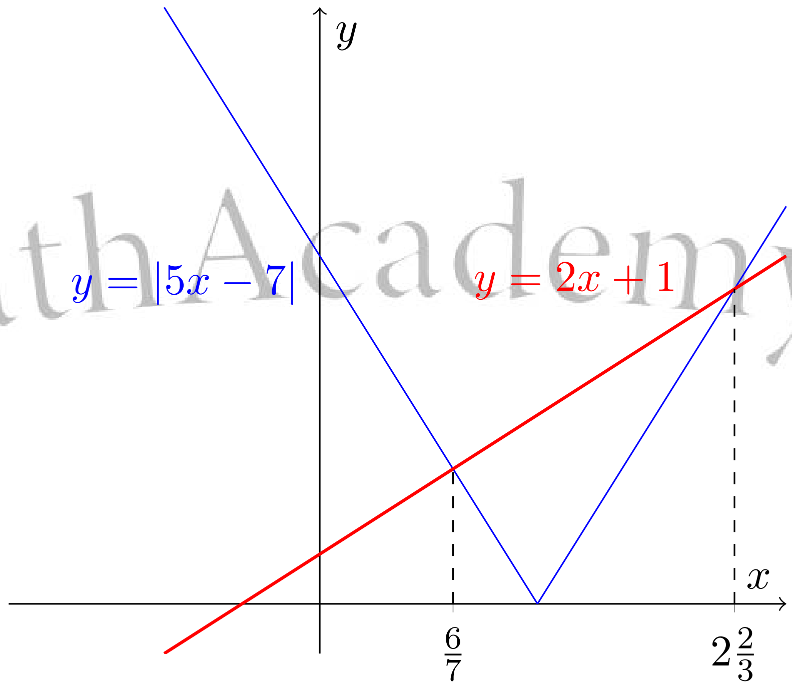
Answer: This should be your first approach! Its the fastest way. However, if the question restricts the use of GC, then we have no c[illegible] per previous sections.

Solving inequality using GC

1. Plot the graphs using GC.
2. Find the points of intersection.
3. Determine and conclude the appropriate region.

Example 14.
Solve the inequality $|5x-7| > 2x+1$ using graphical method.

Solution:



Using GC, the x-coordinate of the points of intersections are $x = \frac{6}{7}$ and $x = 2\frac{2}{3}$. Therefore,

$$x < \frac{6}{7} \quad \text{or} \quad x > 2\frac{2}{3}$$

Example 15 (2011/RI/Prelim/I/1a).
Using a graphical approach, solve the inequality

$$\sqrt{x} < 2x.$$

[$x > \frac{1}{4}$]

Example 16 (2009/SRJC/Prelims/I/1).
Solve the following inequalities:

$$|2 - 3x| < -x^2 + 3x + 2$$

[$0 < x < 2$]

