GRAPHING TECHNIQUES II

1 Transformation of Graphs

Along the y-axis.

$y = f(x)$	Resulting Equation	Transformation Description
Replace y by $y - a$	$y - a = f(x)$ $\implies y = f(x) + a$	Translate by a units in the direction of positive y-axis
Replace y by ay	$ay = f(x)$ $\implies y = \frac{1}{a}f(x)$	Scale parallel to the y-axis by a factor of $\frac{1}{a}$
Replace y by $-y$	$-y = f(x)$ $\implies y = -f(x)$	Reflection about the x-axis

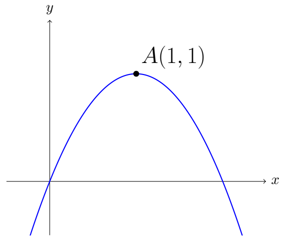
Along the x-axis.

$y = f(x)$	Resulting Equation	Transformation Description
Replace x by $x - a$	$y = f(x - a)$	Translate by a units in the direction of positive x-axis
Replace x by ax	$y = f(ax)$	Scale parallel to the x-axis by a factor of $\frac{1}{a}$
Replace x by $-x$	$y = f(-x)$	Reflection about the y-axis

All transformations should be seen as either a replacement of x or y!

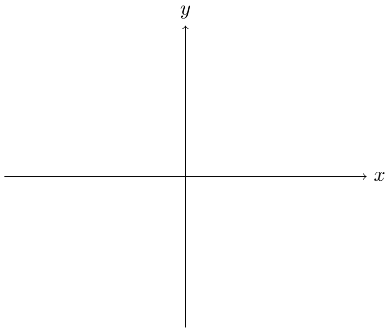
Example 1.

Let $y = f(x)$ be the graph shown on the right.

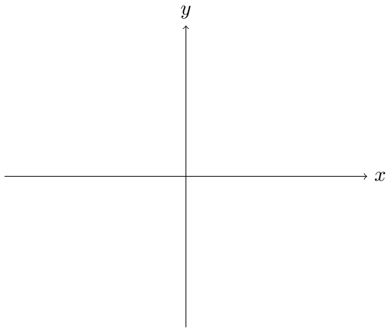


Draw the graph of the following functions, indicating the coordinate of the image of A in each case.

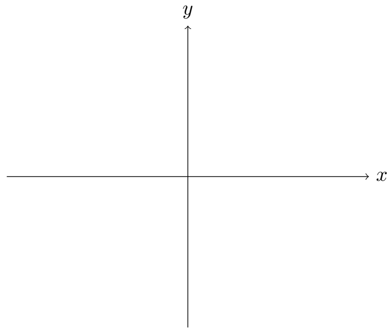
$y - 1 = f(x)$



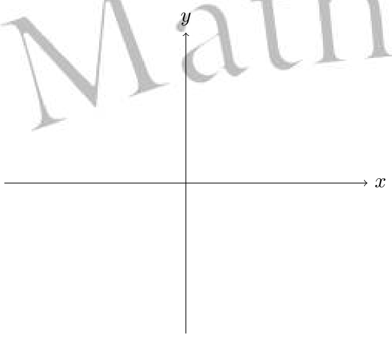
$\frac{y}{2} = f(x)$



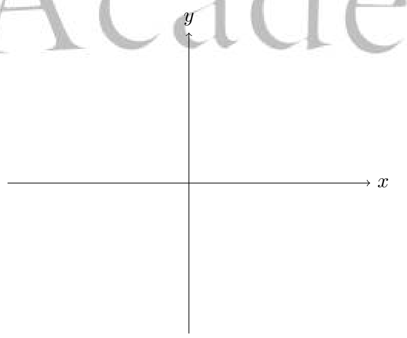
$-y = f(x)$



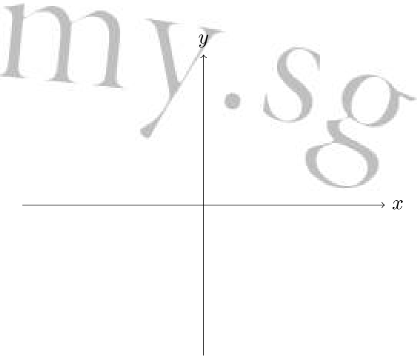
$y = f(x + 1)$



$y = f(2x)$



$y = f(-x)$



1.1 Composite Transformations

Q: How to obtain the graph of $y = f(2x+3)$ from $y = f(x)$?

Method 1: Translate then scale.

$y = f(x)$

Replace x by $x+3$.
$\Rightarrow y = f(x+3)$

Replace x by $2x$.
$\Rightarrow y = f(2x+3)$

Method 2: Scale then translate.

$y = f(x)$

Replace x by $2x$.
$\Rightarrow y = f(2x)$

Replace x by $x + \frac{3}{2}$.
$\Rightarrow y = f\left(2(x+\frac{3}{2})\right) = f(2x+3)$

Example 2.
Describe the sequence of transformations to obtain $y = 4f(3x) + 2$ from $y = f(x)$.
Solution:

$$y = 4f(3x) + 2$$
$$y - 2 = 4f(3x)$$
$$\frac{y-2}{4} = f(3x)$$

$y = f(x)$

$y = f(3x)$: Replace x by
Transformation:

$\frac{y}{4} = f(3x)$: Replace y by
Transformation:

$\frac{y-2}{4} = f(3x)$: Replace y by
Transformation:

Example 3 (Direct transformation).
Describe a sequence of transformations which transform the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2x+3}{x+1}$.

Solution:

$$\begin{aligned} y &= \frac{2x+3}{x+1} \\ &= \frac{2(x+1)+1}{x+1} \\ &= 2 + \frac{1}{x+1} \\ \implies y - 2 &= \frac{1}{x+1} \end{aligned}$$

Example 4 (2011/YJC/Prelim/I/6a).
Describe a sequence of transformations which transform the graph of $y = \frac{3x+4}{x-2}$ to the graph of $y = \frac{1}{x}$.

Example 5 (Reverse transformation).
[2011/PJC/Prelim/I/10]

A graph with the equation $y = f(x)$ undergoes, in succession, the following transformations:

A: A translation of 1 unit in the direction of the x-axis.

B: A stretch parallel to the x-axis by a scale factor $\frac{1}{2}$.

C: A reflection in the y-axis.

The equation of the resulting curve is $y = \frac{4}{4x^2+4x+1}$.

Determine the equation of the graph $y = f(x)$, giving your answer in the simplest form.

Solution:

C': Reflection in the y-axis:
Before C:

$$y = \frac{4}{4(-x)^2 + 4(-x) + 1} = \frac{4}{4x^2 - 4x + 1}$$

B': Stretch parallel to x-axis by a scale factor of 2:
Before B:

$$y = \frac{4}{4\left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right) + 1} = \frac{4}{x^2 - 2x + 1}$$

A': Translation of -1 unit in the direction of x-axis:
Before A:

$$y = \frac{4}{(x+1)^2 - 2(x+1) + 1} = \frac{4}{x^2}$$

Example 6 (2010/CJC/Prelim/I/6).
A graph with equation $y = g(x)$ undergoes in succession, the following transformations:

A: A reflection in the y-axis

B: A translation of 4 units in the direction of the positive x-axis

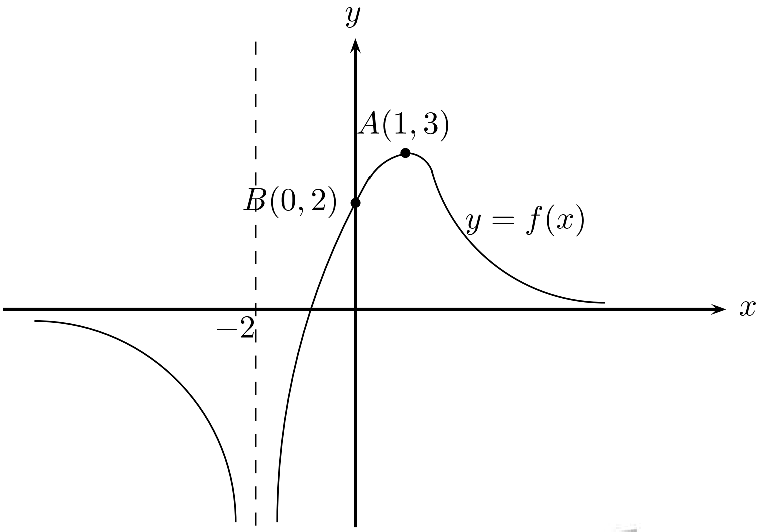
C: Scaling parallel to the y-axis by a factor of 2

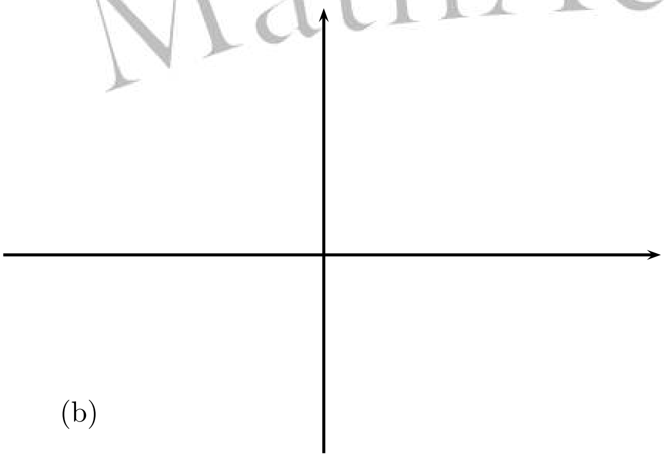
The equation of the resulting curve is $y = 2^{x-3}$. Determine the equation $y = g(x)$. [3]

[$y = 2^{-x}$]

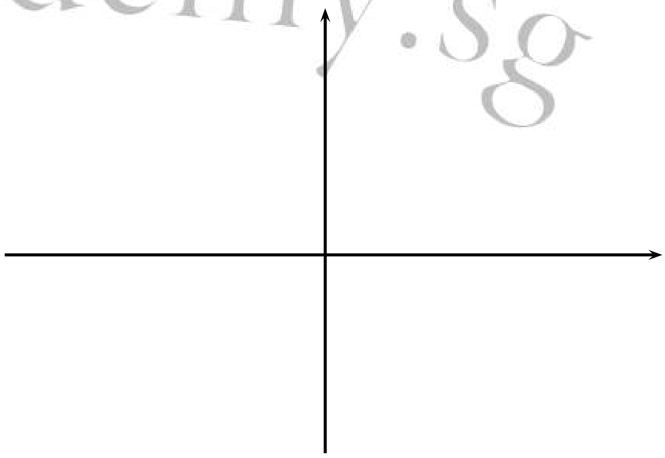
Example 7.
The graph $y = f(x)$ is shown. The coordinates of points A and B are (1,3) and (0,2) respectively. The graph has a vertical asymptote at $x = -2$ and horizontal asymptote at $y = 0$. Sketch on a separate diagram, the graphs of the following, giving the coordinates of the images of A and B in each case.

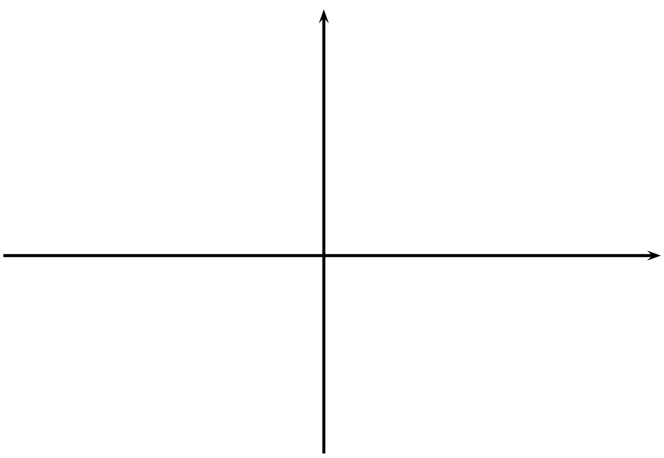
a) $y = -f(x) - 1$ b) $y = f\left(\frac{1}{2}x + 2\right)$ [b) $A'(-2, 3), B'(-4, 2)$]

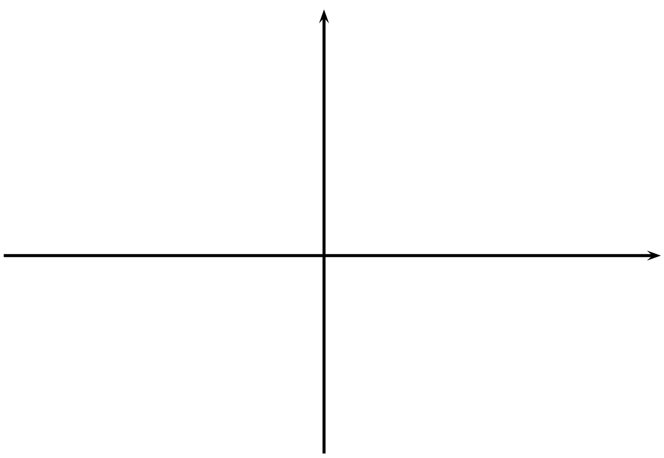




(b)

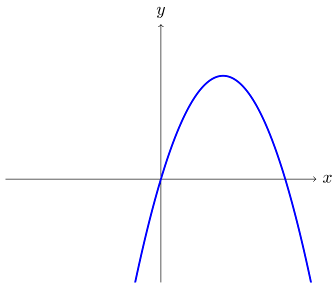




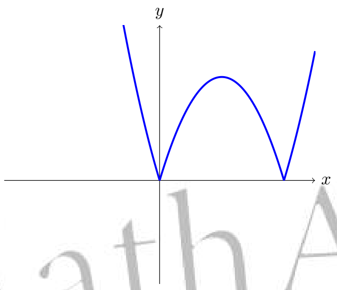


1.2 Modulus Graph

Let $y = f(x)$ be the following function:

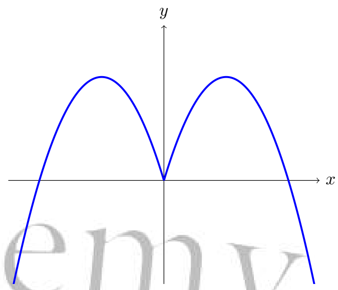


$$|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) < 0 \end{cases}$$



Reflect all parts below x-axis to above.

$$f(|x|) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0 \end{cases}$$



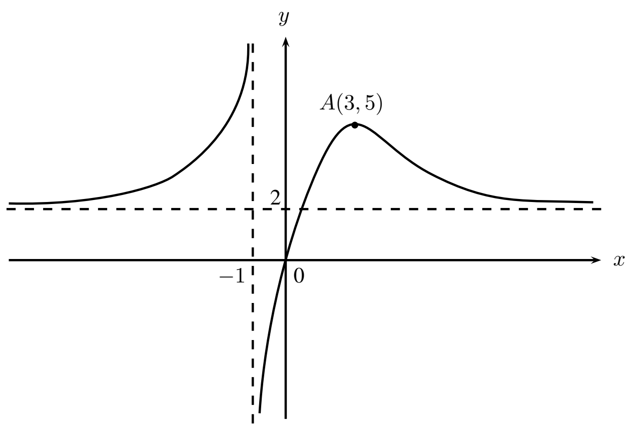
Delete graph to the left of y-axis.
Retain graph to the right of y-axis and reflect to the left.

Example 8.
Given that $y = f(x)$ is as shown, sketch, on separate graphs, the graphs of

(i) $y = f(|x| - 1)$,

(ii) $y = f(|x - 1|)$.

Your sketch should show clearly the equations of any asymptotes and the coordinates of the points corresponding to A.



1.3 Reciprocal graphs $y = \frac{1}{f(x)}$

Step 1. Asymptote:

If $y = c, c \neq 0$ is a horizontal asymptote for $y = f(x)$, then $y = \frac{1}{c}$ is a horizontal asymptote for $y = \frac{1}{f(x)}$.

If $x = a$ is a vertical asymptote for $y = f(x)$, then $x = a$ is a root for $y = \frac{1}{f(x)}$.

Step 2. Root:

If $x = a$ is a root for $y = f(x)$, then $x = a$ is a vertical asymptote for $y = \frac{1}{f(x)}$.

Step 3. Stationary points:

If (a, b) is a max (min) point for $y = f(x)$, then $(a, \frac{1}{b})$ is a min (max) point for $y = \frac{1}{f(x)}$.

Step 4. Behavior at 0 and ∞:

$$\text{If } f(x) \longrightarrow 0, \qquad \text{then } \frac{1}{f(x)} \longrightarrow \infty.$$
$$\text{If } f(x) \longrightarrow \infty, \qquad \text{then } \frac{1}{f(x)} \longrightarrow 0.$$

Note:

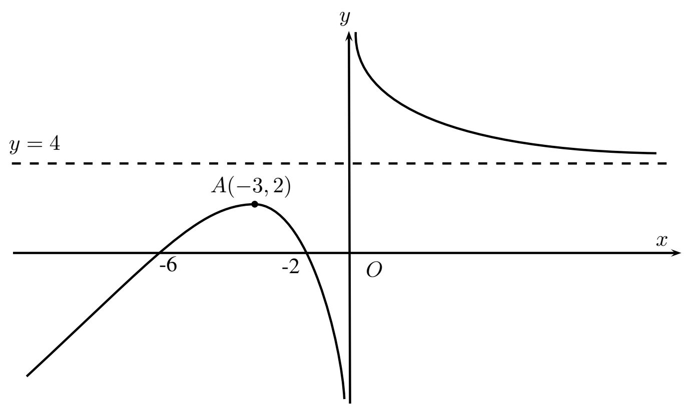
$$f(x) > c \implies \frac{1}{f(x)} < \frac{1}{c}.$$

Therefore, when $y = f(x)$ is above (below) the horizontal asymptote $y = c$, then $y = \frac{1}{f(x)}$ is below (above) the horizontal asymptote $y = \frac{1}{c}$. This is true for $c \neq 0$.

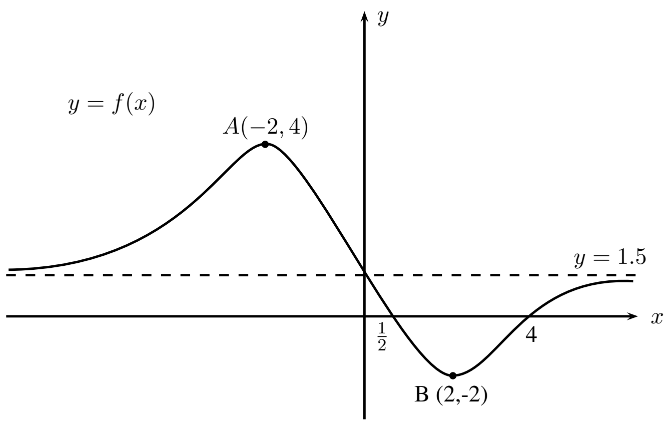
Example 9.

Given that $y = f(x)$ is as shown, sketch, on a separate graph, the graph of $y = \frac{1}{f(x)}$. Your sketch should show clearly the equations of any asymptotes and the coordinates of the points corresponding to A and B.

(a)

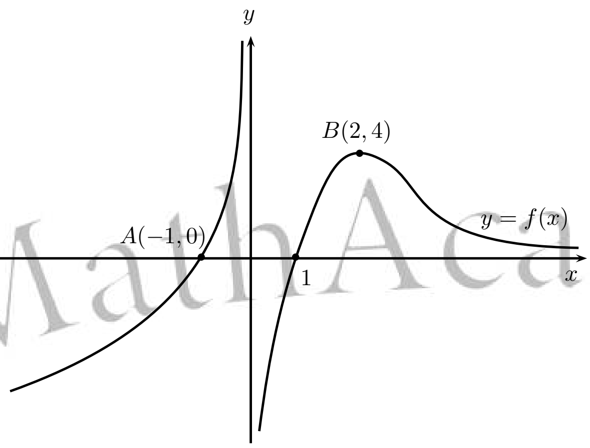


(b)



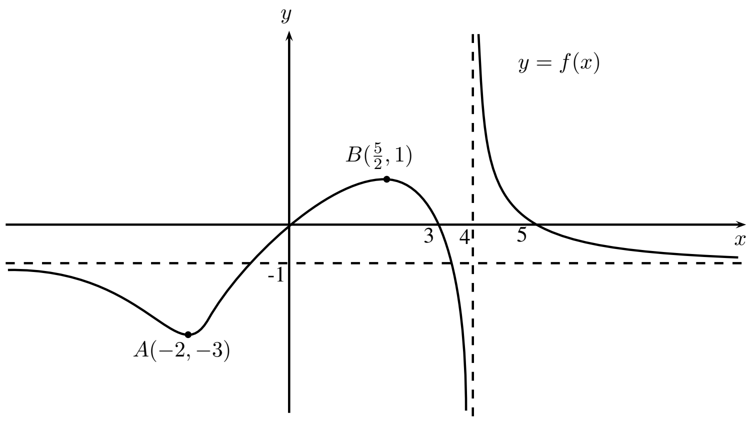
y
$y = f(x)$
$A(-2, 4)$
$y = 1.5$
x
$\frac{1}{2}$
4
B (2,-2)

(c)



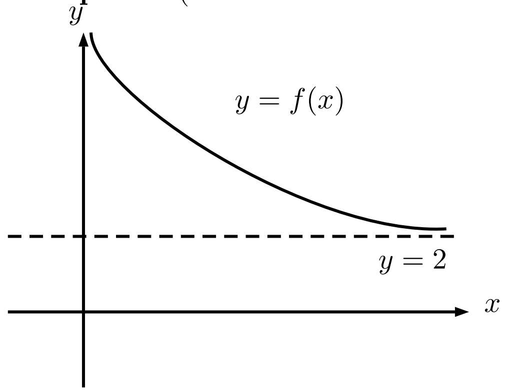
y
$B(2, 4)$
$y = f(x)$
$A(-1, 0)$
1
x

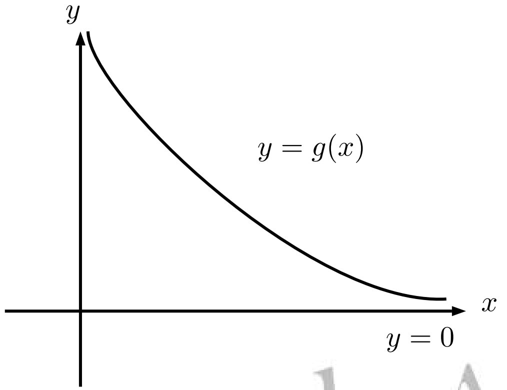
(d)



y
$y = f(x)$
$B(\frac{5}{2}, 1)$
3
4
5
x
-1
$A(-2, -3)$

Example 10 (Difference between horizontal asymptotes).





Example 11.

Given that $y = f(x)$ is as shown, sketch, on a separate graph, the graph of $y = \frac{1}{f(2x)}$. Your sketch should show clearly the equations of any asymptotes and the coordinates of the points corresponding to A.