1 1-1 Functions

Use horizontal line tests to test if a function is 1-1.

(a) If the function is 1-1, say the following [all lines in range]:
The function is 1-1 since ALL horizontal lines $y = k, k \in R_f$, cuts the graph at most once.

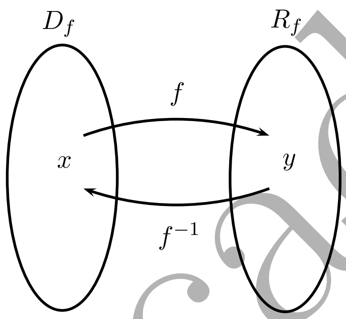
(b) If the function is NOT 1-1, say the following [Find a specific line from your graph]:
The line $y = 3$ (just an example) cuts the graph at 2 points, hence the function is not 1-1.

Alternatively, to show that a function is 1-1 using differentiation, we show that

$$\begin{aligned} & f'(x) > 0 \qquad \text{for all } x \in D_f. \\ \text{or} \quad & f'(x) < 0 \qquad \text{for all } x \in D_f. \end{aligned}$$

Note that differentiation cannot be used to show that a function is NOT 1-1.

2 Inverse Functions



(a) To determine if f^{-1} to exist, we check if f is a 1-1 function.

$$f^{-1} \text{ exists} \iff f \text{ is a 1-1 function}$$

(b) $D_{f^{-1}} = R_f$.

(c) $R_{f^{-1}} = D_f$.

(d) $(f^{-1})^{-1} = f$.

Geometrical relationship between a function and its inverse

(i) The graph of f^{-1} is the reflection of the graph f about the line $y = x$.

(ii) (a, b) lies on $f \Leftrightarrow (b, a)$ lies on f^{-1}.

(iii) $x = k$ is an asymptote of $f \Leftrightarrow y = k$ is an asymptote of f^{-1}

3 Composite Functions

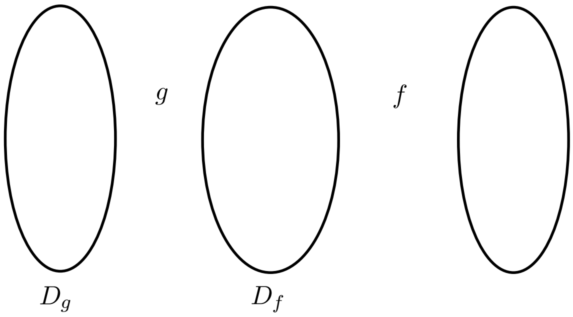


Diagram for function $fg(x)$

(i) Domain fg = domain g.

(ii) Composite function fg exists $\Leftrightarrow R_g \subseteq D_f$

(iii) To find R_{fg},

Step 1. Find R_g.

Step 2. Sketch the graph f. Use R_g as the x-axis. (i.e. Use R_g as domain of your graph)

Step 3. Find the new range of f.

(iv)

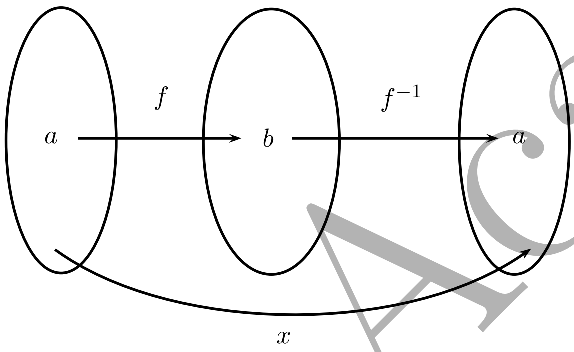


Diagram for function $f^{-1}f(x)$

$ff^{-1}(x) = f^{-1}f(x) = x$

$D_{ff^{-1}} = D_{f^{-1}}$.
$D_{f^{-1}f} = D_f$.

The 2 functions $ff^{-1}(x)$ and $f^{-1}f(x)$ have different domains, this is important when you are asked to sketch the graph of either ff^{-1} or $f^{-1}f$. You must sketch them on the correct domain.

Example 1 (2009/SAJC/Prelim/II/4)**.**
The functions f and g are defined by

$$f : x \to 5 - x^2, x < k$$

$$g : x \to |3 + x|, x \in \mathbb{R}$$

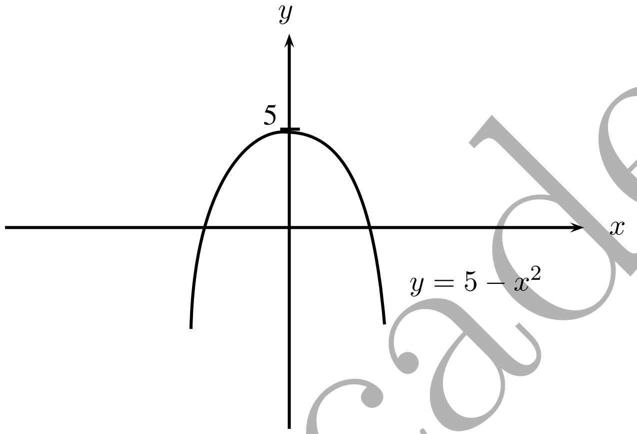
(a) Given that the inverse f^{-1} exists, find the largest value of k and define f^{-1} in similar form.

(b) Sketch, on the same diagram, the graphs of f, f^{-1} and ff^{-1}. Solve the equation $f(x) = f^{-1}(x)$, giving your answer(s) in exact form.

(c) Determine whether the composite functions gf^{-1} exists. Find its rule and domain if it exists.

Solution:

(a)

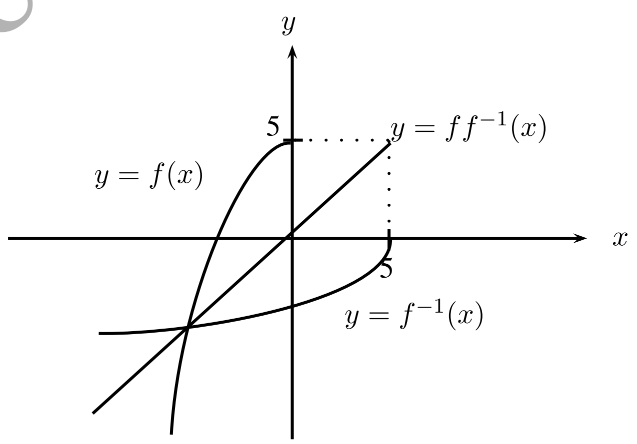


From the graph of $y = 5 - x^2$, largest value of $k = 0$. To find its inverse:

$$\begin{aligned} y &= 5 - x^2 \\ x^2 &= 5 - y \\ x &= \sqrt{5-y} \quad \text{or} \quad x = -\sqrt{5-y} \\ &\text{(rej as } x < 0) \end{aligned}$$

$$\therefore f^{-1}(x) = -\sqrt{5-x}, \quad x < 5.$$

(b)



Since the graphs f and f^{-1} intersect at $y = x$, solving $f(x) = f^{-1}(x)$ is equivalent to solving:

$$f(x) = x$$
$$5 - x^2 = x$$
$$x^2 + x - 5 = 0$$
$$x = \frac{-1-\sqrt{21}}{2} \quad \text{or} \quad x = \frac{-1+\sqrt{21}}{2}$$
$$\text{(rej)}$$

What about $f(x) \leq f^{-1}(x)$ or $f^{-1}(x) \leq f(x)$?

(c) To check if gf^{-1} exists, we want to check if $R_{f^{-1}} \subseteq D_g$,

$R_{f^{-1}} = D_f = (-\infty, 0)$
$D_g = \mathbb{R}$

$(-\infty, 0) \subseteq \mathbb{R} \implies R_{f^{-1}} \subseteq D_g$, hence, gf^{-1} exists.

$$\begin{aligned} gf^{-1}(x) &= g(-\sqrt{5-x}) \\ &= |3 - \sqrt{5-x}|, \qquad x < 5 \end{aligned}$$

Example 2.
The function f is defined by

$$f : x \to \frac{2-3x}{3-x}, \quad x > 3,$$

Solve $f^{-1}(2)$.

Example 3.
Consider the following functions:

$$fg : x \mapsto x^2 - 3x + 2 \qquad x \in \mathbb{R}.$$
$$g : x \mapsto x + 2, \qquad x > 0.$$

Find the function f.

Example 4 (2018/DHS/Prelim/I/5).
Functions f and g are defined by

$$f : x \mapsto 1 + \frac{2}{x-1}, \quad x \in \mathbb{R}, x < 1,$$
$$g : x \mapsto \ln x, \quad x \in \mathbb{R}, 0 < x < 1.$$

(i) Explain why the composite function gf does not exist. [1]

(ii) Find the function $fg(x)$. Hence or otherwise, find $(fg)^{-1}(0)$. [4]

(iii) Find an expression for $h(x)$ for each of the following cases:

(a) $gh(x) = x$, [1]

(b) $hg(x) = x^2 + 1$. [2]

[(ii) $fg(x) = \frac{\ln x+1}{\ln x-1}, \quad x \in \mathbb{R}, 0 < x < 1.$; (ii) $(fg)^{-1}(0) = e^{-1}$ (iii)a) $h(x) = e^x$ (b) $h(x) = e^{2x} + 1$]

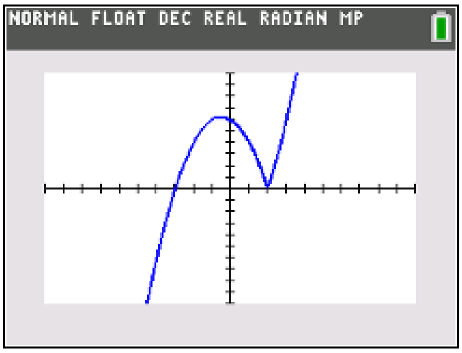
Example 5 (2018/AJC/Prelim/I/9ii).

$$f : x \mapsto (x+3)|x-2|, \quad x \in \mathbb{R}.$$

The domain of f is restricted to $[k, 2]$ where k is the least value such that f^{-1} exists.

Write down the value of k. Find $f^{-1}(x)$ and state the domain of f^{-1}.

$y = f(x)$



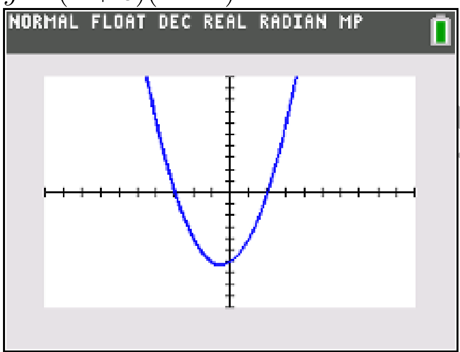
From the graph on the GC, $k = -0.25$.

Removing modulus for functions question

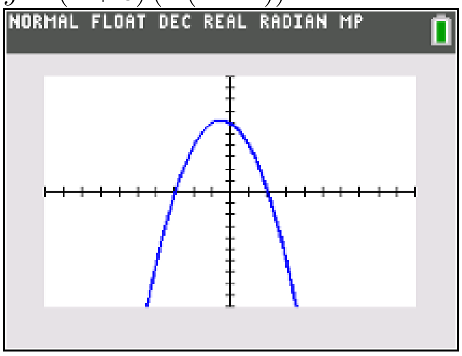
We draw the 2 graphs on the GC, the plus, and the minus, after the modulus is removed.

To know which equation to pick, we choose the one that coincides with our original graph, **on the domain that we are interested in**.

$y = (x+3)(x-2)$



$y = (x+3)\left(-(x-2)\right)$



Example 6 (2010/TPJC/Prelim/I/10).
The function f and g are defined by

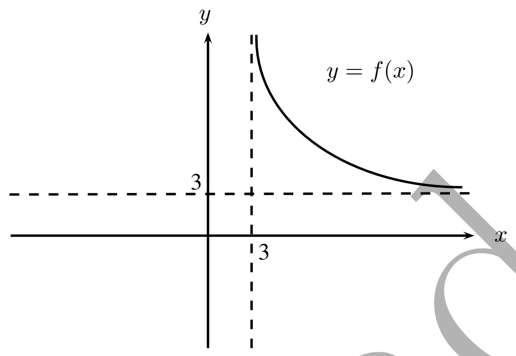
$$f : x \to \frac{2-3x}{3-x}, \quad x > 3,$$

$$g : x \to x^2 - 4x + \alpha, \quad \alpha \in \mathbb{R}, x > 1.$$

(a) Define, in a similar manner, the inverse function f^{-1} and show that $f^2(x) = x$. Hence determine f^{13} in a similar manner.

Solution:

(a) $y = \frac{2-3x}{3-x} = \frac{3(3-x)-7}{3-x} = 3 - \frac{7}{3-x}$



From the graph, $R_f = (3, \infty)$. So, $D_{f^{-1}} = (3, \infty)$.

$$\begin{aligned} (3-x)y &= 2-3x \\ 3y - xy - 2 + 3x &= 0 \\ x(-y+3) &= 2-3y \\ x &= \frac{2-3y}{3-y} \end{aligned}$$

$$\therefore f^{-1}(x) = \frac{2-3x}{3-x}, \qquad x \in (3, \infty)$$

Therefore,

$$\begin{aligned} f(x) &= f^{-1}(x) \\ ff(x) &= ff^{-1}(x) \\ f^2(x) &= x \end{aligned}$$

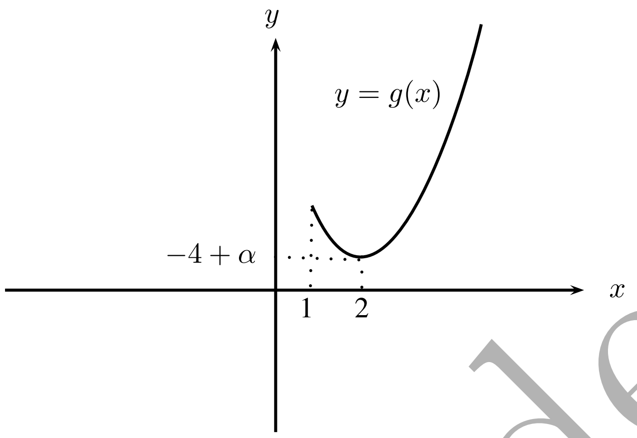
$$\begin{aligned} f^{13}(x) &= f[f^{12}(x)] \\ &= f(x) && \text{(since } f^{12}(x) = x) \\ &= \frac{2-3x}{3-x}, \qquad x \in (3, \infty) \end{aligned}$$

(b) Find the range of g, in terms of α. For $\alpha = 8$, find the range of fg.

Solution:

To find R_g, we complete the square to obtain its min point.

$$\begin{aligned} y &= x^2 - 4x + \alpha \\ &= (x-2)^2 - 4 + \alpha \end{aligned}$$



$\therefore R_g = [-4+\alpha, \infty)$

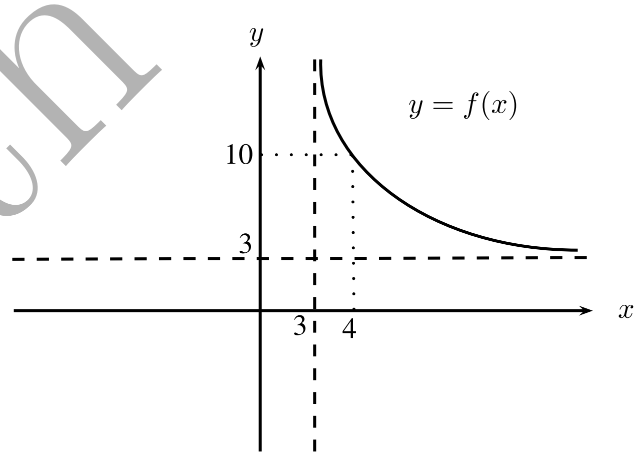
[To find R_{fg}]

Step 1. Find R_g.

Step 2. Use R_g as the x-axis for the graph of f. (i.e. Use R_g as D_f)

Step 3. Find the new range of f.

When $\alpha = 8, R_g = [4, \infty]$.



$$f(4) = \frac{2-3(4)}{3-4} = 10.$$

We use R_g on the x-axis on the graph of f. Under this new domain, $R_{fg} = (3, 10]$.

Example 7 (2010/RI/II/3modified)**.**
Functions f and g are defined by

$f : x \to (x-2)(x-4)$, for $x \in \mathbb{R}, x < 4$,

$g : x \to \frac{x}{x+3}$, for $x \in \mathbb{R}\backslash\{-3\}$.

i. Determine if f^{-1} exists, justifying your answer. [2]

ii. Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist.

Find the range of the composite that exists. [6]

$\left[\text{(ii)}[-\frac{1}{2}, 1)\right]$

Piecewise Functions

Example 8 (2014/PJC/Prelim/I/7modified).
It is given that

$$f(x) = \begin{cases} \frac{1}{1+\sqrt{x}} & \text{for } 0 \le x < 4, \\ -1 & \text{for } 4 \le x < 5, \end{cases}$$

and that $f(x+5) = f(x)$ for all real values of x.

(i) Find $f(24)$ and $f(30)$.

(ii) Sketch the graph of $y = f(x)$ for $-5 \le x < 12$.

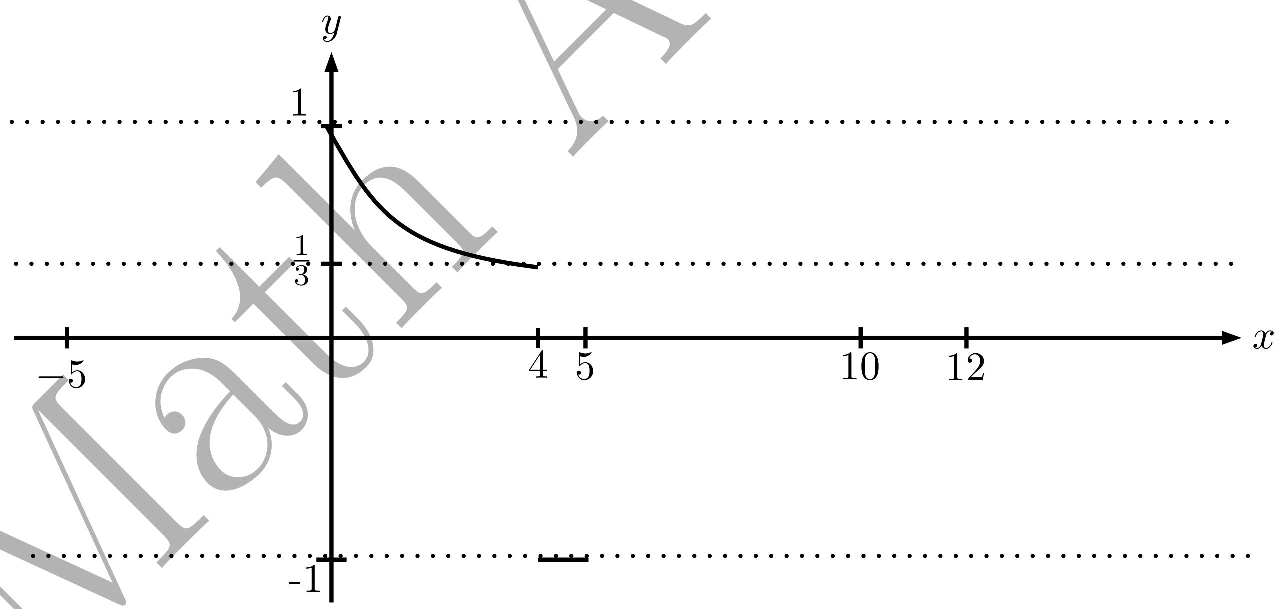
Solution:

(i)

$$\begin{aligned} f(24) &= f(19) \\ &= f(14) \\ &= f(9) \\ &= f(4) \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(30) &= f(25) \\ &= f(20) \\ &= f(15) \\ &= f(10) \\ &= f(5) \\ &= f(0) \\ &= \frac{1}{1+\sqrt{0}} = 1 \end{aligned}$$

(ii)



Piecewise Inverse

Example 9 (2018/HCI/II/3)**.**
The function f is defined as

$$f(x) = \begin{cases} ae^{a-x} & \text{for } 0 \leq x < a, \\ \left|\frac{1}{a}(x-a)^2 - a\right| & \text{for } x \geq a. \end{cases}$$

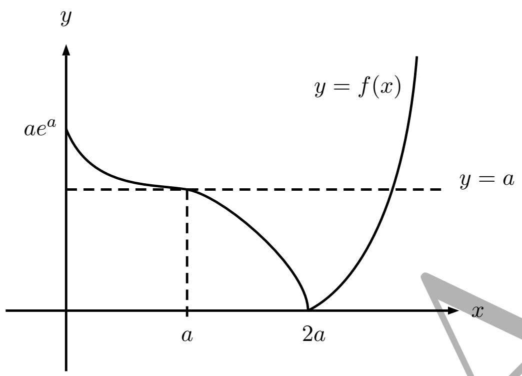
(i) Sketch the graph of $y = f(x)$, indicating clearly the axial intercepts. Show that f^{-1} does not exist.

(ii) If the domain of f is restricted to $[0, k]$, determine the largest value of k in terms of a such that f^{-1} exists.

(iii) Using the domain found in part (ii), define f^{-1} in similar form.

Solution:

(i)



Since the line $y = a$ cuts the graph $y = f(x)$ more than once, f is not a one-one function and therefore f^{-1} does not exist.

(ii) Largest value of $k = 2a$.

(iii)

Piecewise Composite

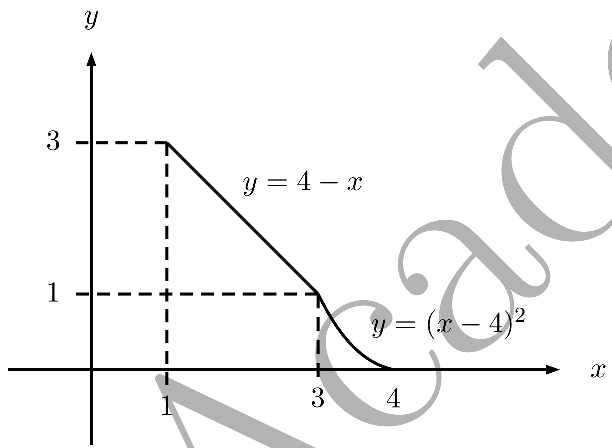
Example 10 (2018/EJC/II/4)**.**
The functions g and f are defined by

$$g(x) = \begin{cases} 4 - x & \text{for } 1 \le x < 3, \\ (x-4)^2 & \text{for } 3 \le x < 4. \end{cases}$$

$$h(x) = \begin{cases} \sqrt{1-x} & \text{for } 0 \le x < 1, \\ (x-1)^4 & \text{for } 1 \le x \le 3. \end{cases}$$

Given that hg exist, define hg in similar form as function h.

Solution:



Example 11 (2018/MJC/Prelim/II/4b).
The function f is defined by

$$f(x) = \begin{cases} (x+1)^2 + a & \text{for } -1 < x \leq 1, \\ (4+a)(2-x) & \text{for } 1 < x \leq 2, \end{cases}$$

where a is a positive real constant and that $f(x) = f(x+3)$ for all real values of x.

(i) Evaluate $f(-41)$ and $f(2018)$. [2]

(ii) Sketch the graph of $y = f(x)$ for $-4 < x \leq 6$. [3]

(iii) Hence find the value of $\int_{-4}^{4} f(x)\mathrm{d}x$ in terms of a. [3]

[(i) $f(-41) = 4 + a$; $f(2018) = 0$ (iii) $12 + 7a$]