Functions Revision

1. [2016/HCI/II/11]

   The functions f and g are defined by

   $$f : x \mapsto \frac{1}{2}e^{1-x^2}, x \in \mathbb{R}, x \leq 1 \text{ and}$$

   $$g : x \mapsto \sqrt{1 - \ln x}, x \in \mathbb{R}, 0 < x \leq e.$$

   i. Show that gf exists, and find the range of gf. [4]

   ii. Justify, with a reason, whether f^{-1} exists. [2]

   iii. The domain of f is restricted to $(-\infty, b]$ such that b is the largest value for which the inverse function f^{-1} exists. State the value of b and define f^{-1} clearly. [4]

   iv. The graph of $y = h(x)$ is obtained by transforming the graph of $y = g(x)$ in the following 2 steps:

      Step 1: Scale the parallel to the x-axis by a factor of 2.

      Step 2: Reflect in the x-axis.

      Define h in a similar form. [3]

   [(i) $R_{gf} = [\sqrt{\ln 2}, \infty)$ (iii) $b = 0$; $f^{-1} : x \mapsto -\sqrt{1 - \ln(2x)}, x \in \mathbb{R}, 0 < x \leq \frac{1}{2}e$ (iv) $h : x \mapsto -\sqrt{1 - \ln\left(\frac{x}{2}\right)}, x \in \mathbb{R}, 0 < x \leq 2e$]

2. [2016/JJC/I/6]

   A function f is said to be self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.

   The function g is defined by

   $$g : x \mapsto \sqrt{\frac{x^2+2}{x^2-1}}, x > 1.$$

   i. Sketch the curve $y = g(x)$, stating the equations of the asymptotes clearly. [2]

   ii. Define g^{-1} in a similar form and show that g is self-inverse. [4]

   iii. Show that $g^2(x) = x$ and that $g^3(x) = g(x)$. Hence find the values of x for which [4]

   $$4 - g^{50}(x) = [g^{51}(x)]^2.$$

   [(ii) $g^{-1} : x \mapsto \sqrt{\frac{x^2+2}{x^2-1}}, x > 1$ (iii) $x = 2$ or $x = \frac{1+\sqrt{13}}{2}$]

3. [2016/NJC/II/2]

   The functions f and g are defined by

   $$f : x \mapsto x^2 - 4x + 3, \text{ for } x \in \mathbb{R}, x \leq a \text{ and}$$

   $$g : x \mapsto \tan^{-1}(2x+1), \text{ for } x \in \mathbb{R}, x > -2,$$

   where a is a constant.

   (a) If $a = 2$, solve the equation $f(x) = x$ exactly. [2]

   (b) If $a = 3$,

      i. give a reason why f has no inverse. [2]

      ii. Prove that the composite function gf exists and state the rule, domain and exact range of the composite function. [6]

[(a) $\frac{5-\sqrt{13}}{2}$ (b)ii) $gf : x \mapsto \tan^{-1}(2x^2 - 8x + 7), x \in \mathbb{R}, x \leq 3$; $R_{gf} = \left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$]

4. [2016/NYJC/I/10]

   (a) i. Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$ where R and α are exact positive constants to be found. [1]

   The function f is defined by $f : x \mapsto \sin x + \sqrt{3}\cos x, \frac{\pi}{6} \leq x \leq k$.

   ii. Find the largest exact value of k such that f has an inverse. Hence define f^{-1} in similar form and write down the set of values of x for which $ff^{-1}(x) = f^{-1}f(x)$. [5]

   (b) The function g is defined by $g : x \mapsto 2 - \frac{5x}{1+x^2}, x \in \mathbb{R}$.

   i. Use an algeraic method to find the range of g. [3]

   ii. State a sequence of transformations which transforms the graph of $y = g(x)$ to the graph of $y = \frac{10x}{4+x^2}$. [3]

   [a)ii) $k = \frac{7\pi}{6}$; $\{x \in \mathbb{R} : \frac{\pi}{6} \leq x \leq 2\}$ (b)i) $R_g = \left[-\frac{1}{2}, \frac{9}{2}\right]$]

5. [2016/PJC/II/3]

   The function f is defined by $f(x)=\begin{cases} \frac{x}{2} & \text{if } x \leq 0, \\ 2\sin x & \text{if } 0 < x \leq 4. \end{cases}$

   i. Sketch the graph of $y = f(x)$. [2]

   ii. If the domain of f is restricted to $x \leq k$, state the largest value of k, in exact form, for which the function f^{-1} exist. [1]

   iii. Using the domain from part (ii), define f^{-1} in a similar form. [4]

   iv. Solve $f^{-1}(x) = f(x)$. [2]

   In the rest of the question, the domain of f is as originally defined.

   The function g is defined by $g : x \mapsto -x^3, x \in \mathbb{R}, x > 0$.

   v. Find an expression for $fg(x)$. [2]

   [(ii)$k = \frac{\pi}{2}$ (iii)$f^{-1}(x)=\begin{cases} 2x & \text{if } x \leq 0, \\ \sin^{-1}\left(\frac{x}{2}\right) & \text{if } 0 < x \leq 2. \end{cases}$ (iv) $x = 0$ (v) $fg(x) = \frac{-x^3}{2}$]

6. [2011/NJC/Prelim/I/7]

   The function f is defined by $f : x \mapsto 1 - (x + 3)^2,\ x \in \mathbb{R},\ x \geq -2$.

   (a) Show that the inverse function of f exists. Find $f^{-1}(x)$ and state its domain. [4]

   (b) On the same diagram, sketch the graphs of $y = f(x), y = f^{-1}(x)$ and $y = f^{-1}f(x)$, showing clearly the relationship between the graphs. [3]

   (c) Hence, find the exact solution of $f^{-1}f(x) \leq f(x)$. [3]

   [(a) $-3 + \sqrt{1 - x}, x \leq 0$ (c) $-2 \leq x \leq \frac{-7+\sqrt{17}}{2}$]

7. [2011/HCI/Prelim/II/2]

   (a) The function is defined by

   $$f : x \mapsto x^2 + \lambda x + 2, \quad x \in \mathbb{R}, x \leq 2,$$

   where λ is a constant.

   i. Find the range of values of λ such that f^{-1} exists. [2]

ii. Given that $\lambda = -4$, obtain f^{-1} in a similar form. [3]

(b) The functions g and h are defined by

$$g : x \mapsto x^2 + kx + 1, \quad x \in \mathbb{R} \text{ and } k \text{ is a constant,}$$

$$h : x \mapsto x + \frac{4}{x}, \quad x \in \mathbb{R}, x > 0.$$

i. Find the range of values of k such that hg exists. [3]

ii. Given that $k = 1$, find the range of hg. [2]

[a(i) $\lambda \leq -4$ a(ii) $2 - \sqrt{x+2}, x \geq -2$ b(i) $-2 < k < 2$ b(ii) $[4, \infty)$]

8. [2017/VJC/I/3]

It is given that

$$f(x) = \begin{cases} (x-2)^2 - 1, & \text{for } 0 < x \leq 3, \\ x - 3, & \text{for } 3 < x \leq 6, \end{cases}$$

and that $f(x) = f(x+6)$ for all real values of x.

i. Sketch the graph of $y = f(x)$ for $0 < x \leq 10$. [3]

ii. On a separate diagram, sketch the graph of $y = 1 + f\left(\frac{1}{2}x\right)$ for $0 < x \leq 10$. [2]

9. [2015/SAJC/Prelim/I/9]

The function g, with domain $\{x \in \mathbb{R} : 1 \leq x \leq 6\}$, has six of its function values given in the table below.

x	1	2	3	4	5	6
$g(x)$	4	5	1	3	2	5

i. Use the table to explain why g does not have an inverse function. [2]

ii. Find $g^3(3)$. Hence find the set of all positive integers n for which $g^n(3) = 4$. [3]

It is also known that for $k = 1, 2, 3, 4, 5$

$$g(x) = g(k) + (g(k+1) - g(k))(x - k) \quad \text{for } k < x < k + 1.$$

iii. Evaluate $g(1.5)$ and $g(2.7)$. [2]

iv. Sketch the graph $y = g(x)$. [2]

v. Find the range of values of k for which the equation $g(x) = k$ has four real distinct roots. [1]

[(ii) $\{n \in \mathbb{Z}^+ : n = 3k - 1\}$ (iii) $g(1.5) = 4.5$, $g(2.7) = 2.2$ (v) $2 < k < 3$]