Maclaurin's Series Tutorial 2

1. [2014/MJC/Prelim/I/5]

   Given that $y = \cos^{-1}(2x)$, prove that $\sin y \frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 0$. [3]

   i. By further differentiation of this result, find the series expansion of y in ascending powers of x up to and including the term in x^3. Give the coefficients in exact form. [3]

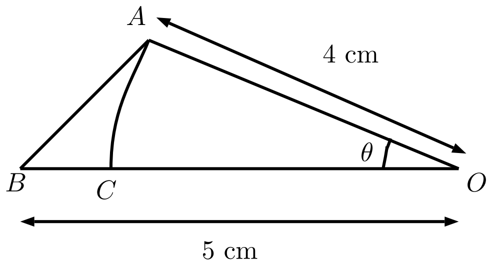
   ii. Hence find the series expansion of $\frac{\cos^{-1}(2x)}{\sqrt{1-3x^2}}$ in ascending powers of x up to and including the term in x^2. Give the coefficients in exact form. [2]

   [(i) $y = \frac{\pi}{2} - 2x - \frac{4}{3}x^3 + \ldots$ (ii) $\frac{\pi}{2} - 2x + \frac{3\pi}{4}x^2 + \ldots$]

2. [2015/MJC/Prelim/I/7]

   (a) Given that $f(x) = e^{\sin ax}$, where a is a non-zero real constant, find $f(0)$, $f'(0)$ and $f''(0)$. Hence write down the first three non-zero terms in the Maclaurin series of $f(x)$. Give the coefficients in terms of a. [5]

   (b) [It is given that arc length of a circular sector with radius r and angle θ radius is $r\theta$.]

   The diagram shows a triangle OAB with $OA = 4$cm, $OB = 5$cm and $\angle AOB = \theta$ radians. Given that OA and OC are radii of a circle with centre O and θ is a sufficiently small angle, show that the perimeter of ABC can be approximated by $a + b\theta + c\theta^2$ for constants a, b and c to be determined. [5]

   [(a) $f(0) = 1, f'(0) = a, f''(0) = a^2$ (b) $a = 2, b = 4, c = 10$]

3. [2010/TPJC/Prelim/II/5]

   Given that $y = \ln(1+2x)$, show that $(1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$. [1]

   (a) By repeated differentiation of this result, find the Maclaurin's expansion for y up to and including the term in x^5. [5]

   (b) By writing down the Maclaurin's expansion of $\ln(1-2x)$, show that

   $$\ln\left(\frac{1+2x}{1-2x}\right) = 2\left[2x + \frac{8}{3}x^3 + \frac{32}{5}x^5 + \ldots\right]$$

   [2]

   (c) *Use the series in part (b) for $x = \frac{1}{4}$, find the exact value of $\sum_{r=0}^{\infty} \frac{1}{(2r+1)2^{2r+1}}$. [3]

   [(a) $y = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5 + \ldots$ (c) $\frac{1}{2}\ln 3$]

4. [2010/DHS/Prelim/I/9]

   (a) It is given that $y = \ln(1+e^x)$.

   i. Show that $\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 - \frac{dy}{dx})$. [3]

   ii. Find the Maclaurin's series for y up to and including the term in x^2. [2]

   iii. *Verify that the same result is obtained if the standard series expansions for e^x and $\ln(1+x)$ are used. [4]

   (b) Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that

   $$10\tan x - 3 = \cos 2x,$$

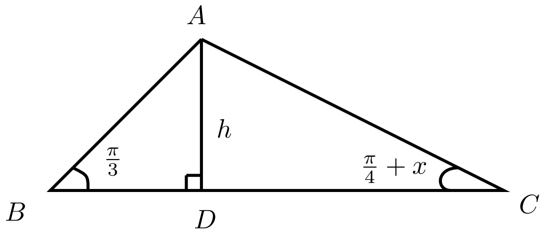
   form a quadratic equation in x and hence find the value of x, leaving your answer in exact form. [3]

   [a(ii) $y = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$ (b) $x = \frac{-5+\sqrt{33}}{2}$]

Hint a)iii): Use the standard series expansion for e^x. Replace e^x in $y = \ln(1+e^x)$ by that standard series.

5. [2017/SAJC/I/2]

The diagram shows the triangle ABC. It is given that the height AD is h units, $\angle ABD = \frac{\pi}{3}$ and $\angle ACD = \frac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC = \frac{h}{\sqrt{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)} \approx h(p + qx + rx^2)$$

for constants p, q, r to be determined in exact form. [5]

$[h\left(1 + \frac{\sqrt{3}}{3} - 2x + 2x^2\right)]$

6. [2013/RI/I/2]

In the triangle PQR, angle $PQR = \left(\frac{\pi}{6} - \theta\right)$ radians, angle $PRQ = \left(\frac{\pi}{6} + \theta\right)$ radians and $QR = 3$. Given that θ is sufficiently small, show that

$$PQ - PR = a\left[\sin\left(\frac{\pi}{6} + \theta\right) - \sin\left(\frac{\pi}{6} - \theta\right)\right] \approx b\theta,$$

for constants a and b to be determined. [5]

7. [2010/MJC/P1/Q4]

(a) Express $f(x) = \frac{x-4}{(x+1)(3x+2)}$ in partial fractions.
Hence, expand $f(x)$ in ascending powers of x, up to and including the term in x^3. [4]

(b) State the range of values for which this expansion is valid. [1]

(c) Find the coefficient of x^n in this expansion. [2]

[(a) $-2 + \frac{11}{2}x - \frac{43}{4}x^2 + \frac{149}{8}x^3 + \ldots$ (b) $-\frac{2}{3} < x < \frac{2}{3}$ (c) $(-1)^n\left[5 - 7\left(\frac{3}{2}\right)^n\right]$]

8. [IJC/2009/Prelims/I/7]

Find the expansion of $\left[\frac{1-x}{1+2x}\right]^{\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^2. State the range of values of x for which the expansion is valid. [5]

(a) Without performing any calculations, explain why putting $x = \frac{1}{6}$ into the result gives a better approximation to $\sqrt[3]{5}$ than putting $x = -\frac{4}{11}$. [1]

(b) Hence by putting $x = \frac{1}{6}$, find an approximation for $\sqrt[3]{5}$, expressing your answer as a fraction in its lowest terms. [2]

$[1 - x + x^2 + \ldots, -\frac{1}{2} < x < \frac{1}{2}$ (b) $\frac{31}{18}]$

Solutions

1. $y = \cos^{-1}(2x) \implies \cos y = 2x$

   Differentiate w.r.t. x:

$$-\sin y\left(\frac{dy}{dx}\right) = 2$$

   Differentiate w.r.t. x:

$$\begin{aligned}-\cos y\left(\frac{dy}{dx}\right)^2 + (-\sin y)\left(\frac{d^2y}{dx^2}\right) &= 0\\ -(2x)\left(\frac{dy}{dx}\right)^2 - \sin y\left(\frac{d^2y}{dx^2}\right) &= 0\\ 2x\left(\frac{dy}{dx}\right)^2 + \sin y\left(\frac{d^2y}{dx^2}\right) &= 0 \text{ (shown)}\end{aligned}$$

   i. Differentiate w.r.t. x:

$$\begin{aligned}2\left[\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx}\frac{d^2y}{dx^2}\right] + \sin y\frac{d^3y}{dx^3} + \cos y\frac{dy}{dx}\frac{d^2y}{dx^2} &= 0\\ 2\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}\frac{d^2y}{dx^2}(2x + \cos y) + \sin y\frac{d^3y}{dx^3} &= 0\end{aligned}$$

   When $x = 0$,

   $\implies y = \cos^{-1} 0 = \frac{\pi}{2}$.

   $\implies -\sin\frac{\pi}{2}\left(\frac{dy}{dx}\right) = 2 \quad \implies -\frac{dy}{dx} = 2 \quad \implies \frac{dy}{dx} = -2.$

   $\implies 0 + \sin\frac{\pi}{2}\left(\frac{d^2y}{dx^2}\right) = 0 \quad \implies \frac{d^2y}{dx^2} = 0.$

   $\implies 2(-2)^2 + 0 + \sin\frac{\pi}{2}\left(\frac{d^3y}{dx^3}\right) = 0 \quad \implies \frac{d^3y}{dx^3} = -8.$

   $\therefore y = \frac{\pi}{2} - 2x + \frac{-8}{3!}x^3 + \ldots = \frac{\pi}{2} - 2x - \frac{4}{3}x^3 + \ldots$

   ii.

$$\begin{aligned}\frac{\cos^{-1}(2x)}{\sqrt{1-3x^2}} &= \cos^{-1}(2x)\cdot(1-3x^2)^{-\frac{1}{2}}\\ &= \left(\frac{\pi}{2} - 2x - \frac{4}{3}x^3 + \ldots\right)\cdot\left(1 + \left(-\frac{1}{2}\right)(-3x^2) + \ldots\right)\\ &= \left(\frac{\pi}{2} - 2x - \frac{4}{3}x^3 + \ldots\right)\cdot\left(1 + \frac{3}{2}x^2 + \ldots\right)\\ &= \frac{\pi}{2} - 2x + \frac{3\pi}{4}x^2 + \ldots\end{aligned}$$

2. (a)

$$f(x) = e^{\sin ax}$$

$$\begin{aligned}f'(x) &= a\cos ax e^{\sin ax}\\ &= a\cos ax f(x)\end{aligned}$$

$$\begin{aligned}f''(x) &= a(\cos ax f'(x) - a\sin ax f(x))\\ &= a\cos ax f'(x) - a^2\sin ax f(x)\end{aligned}$$

$$\begin{aligned}f(0) &= e^0 = 1\\ f'(0) &= a\cos 0 f(0) = a\\ f''(0) &= a(\cos 0)(a) - a^2(\sin 0)(1) = a^2\end{aligned}$$

$$\therefore f(x) = 1 + ax + \frac{a^2}{2}x^2 + \ldots$$

(b)

Using cosine rule,

$$\begin{aligned} AB^2 &= OA^2 + OB^2 - 2(OA)(OB)\cos\theta \\ &= 4^2 + 5^2 - 2(4)(5)\cos\theta \\ &= 41 - 40\cos\theta \end{aligned}$$

$$\begin{aligned} AB &= (41 - 40\cos\theta)^{\frac{1}{2}} \\ &\approx \left[41 - 40\left(1 - \frac{\theta^2}{2} + \dots\right)\right]^{\frac{1}{2}} \\ &= [41 - 40 + 20\theta^2 + \dots]^{\frac{1}{2}} \\ &= (1 + 20\theta^2 + \dots)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}(20\theta^2) + \frac{\frac{1}{2}(-\frac{1}{2})}{2}(20\theta^2)^2 + \dots \\ &\approx 1 + 10\theta^2 + \dots \end{aligned}$$

$$\begin{aligned} \therefore \text{perimeter} &= AB + AC + BC \\ &= 1 + 10\theta^2 + 4\theta + 1 + \dots \\ &= 2 + 4\theta + 10\theta^2 + \dots \end{aligned}$$

3. (a) $y = \ln(1 + 2x)$

$\frac{dy}{dx} = \frac{2}{1+2x}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2(-1)(1+2x)^{-2}(2) \\ &= \frac{-4}{(1+2x)^2} \\ &= \frac{2}{(1+2x)} \cdot \frac{-2}{(1+2x)} \\ &= \frac{dy}{dx} \cdot \frac{-2}{(1+2x)} \end{aligned}$$

$$\therefore (1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\begin{aligned} 2\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} &= 0 \\ 4\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} &= 0 \end{aligned}$$

$$\begin{aligned} 4\frac{d^3y}{dx^3} + (1+2x)\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} &= 0 \\ 6\frac{d^3y}{dx^3} + (1+2x)\frac{d^4y}{dx^4} &= 0 \end{aligned}$$

$$\begin{aligned} 6\frac{d^4y}{dx^4} + (1+2x)\frac{d^5y}{dx^5} + 2\frac{d^4y}{dx^4} &= 0 \\ 8\frac{d^4y}{dx^4} + (1+2x)\frac{d^5y}{dx^5} &= 0 \end{aligned}$$

When $x = 0$,

$y = 0$

$\frac{dy}{dx} = 2$

$\frac{d^2y}{dx^2} = 2(-2) = -4$

$$4\left(-4\right)+\frac{d^3y}{dx^3}=0$$
$$\frac{d^3y}{dx^3}=16$$

$$6\left(16\right)+\frac{d^4y}{dx^4}=0$$
$$\frac{d^4y}{dx^4}=-96$$

$$8\left(96\right)+\frac{d^5y}{dx^5}=0$$
$$\frac{d^5y}{dx^5}=768$$

Therefore,

$$y=0+2x+\frac{-4}{2}x^2+\frac{16}{3!}x^3-\frac{96}{4!}x^4+\frac{768}{5!}x^5+\dots$$
$$=2x-2x^2+\frac{8}{3}x^3-4x^4+\frac{32}{5}x^5+\dots$$

(b)

From mf26, we obtain the following:

$$\ln\left(1+x\right)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5+\dots$$

$$y=\ln\left(1+2x\right)$$
$$=2x-2x^2+\frac{8}{3}x^3-4x^4+\frac{32}{5}x^5+\dots$$
$$y=\ln\left(1-2x\right)$$
$$=2\left(-x\right)-2\left(-x\right)^2+\frac{8}{3}\left(-x\right)^3-4(-x)^4+\frac{32}{5}(-x)^5+\dots$$
$$=-2x-2x^2-\frac{8}{3}x^3-4x^4-\frac{32}{5}x^5+\dots$$
$$\therefore\ln\left(\tfrac{1+2x}{1-2x}\right)=\ln(1+2x)-\ln(1-2x)=2\left(2x+\tfrac{8}{3}x^3+\tfrac{32}{5}x^5+\dots\right)$$

(c) Note that $\sum_{r=0}^{\infty}\frac{1}{(2r+1)2^{2r+1}}=\frac{1}{2}+\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}+\frac{1}{7\cdot 2^7}+\dots$

Sub $x=\frac{1}{4}$ into $\ln\left(\frac{1+2x}{1-2x}\right)=2\left[2x+\frac{8}{3}x^3+\frac{32}{5}x^5+\dots\right]$,

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=2\left(\frac{2}{4}+\frac{8}{3}\left(\frac{1}{4}\right)^3+\frac{32}{5}\left(\frac{1}{4}\right)^5+\dots\right)$$
$$=2\left(\frac{2}{4}+\frac{8}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3+\frac{32}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5+\dots\right)$$
$$=2\left(\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5+\dots\right)$$
$$=2\sum_{r=0}^{\infty}\frac{1}{\left(2r+1\right)2^{2r+1}}$$

$$\therefore\sum_{r=0}^{\infty}\frac{1}{\left(2r+1\right)2^{2r+1}}=\frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$
$$=\frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)$$
$$=\frac{1}{2}\ln 3$$

4. a(i)

$y = \ln(1+e^x)$

$\frac{dy}{dx} = \frac{e^x}{1+e^x}$

$$\frac{d^2y}{dx^2} = \frac{e^x}{1+e^x} + \frac{-e^x}{(1+e^x)^2} \cdot e^x$$
$$= \frac{dy}{dx} + \frac{-dy}{dx} \cdot \frac{e^x}{1+e^x}$$
$$= \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$
$$= \frac{dy}{dx}\left(1 - \frac{dy}{dx}\right)$$

a(ii)

When $x = 0$,

$y = \ln 2$

$\frac{dy}{dx} = \frac{1}{2}$

$\frac{d^2y}{dx^2} = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$

$y = \ln 2 + \frac{1}{2}x + \frac{\frac{1}{4}x^2}{2} = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

a(iii)

From mf26, we obtain expansions for $\ln(1+x)$ and e^x as follows:

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$

$$\ln(1+e^x) = \ln\left(1 + \left(1 + x + \frac{1}{2}x^2 + \ldots\right)\right)$$
$$= \ln\left(2 + x + \frac{1}{2}x^2 + \ldots\right)$$
$$= \ln\left(2\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)\right)$$
$$= \ln 2 + \ln\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)$$
$$= \ln 2 + \left(\frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right) - \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)^2 + \ldots$$
$$= \ln 2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{2}\left(\frac{1}{4}x^2 + \ldots\right) + \ldots \qquad \text{(we ignore terms that are greater than } x^2\text{)}$$
$$= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$$

(b)

Note that for small angle approximation, $\tan x \approx x$.

$$
\begin{aligned}
10\tan x - 3 &= \cos 2x \\
10(x) - 3 &= 1 - \frac{1}{2}(2x)^2 \\
10x - 3 &= 1 - 2x^2 \\
2x^2 + 10x - 4 &= 0 \\
x^2 + 5x - 2 &= 0 \\
x &= \frac{-5 \pm \sqrt{(5)^2 - 4(-2)}}{2} \\
&= \frac{-5 \pm \sqrt{33}}{2}
\end{aligned}
$$

(reject -ve as x is small.)

5. (i)

$$
\begin{aligned}
BC &= BD + DC \\
&= \frac{h}{\tan \frac{\pi}{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)}
\end{aligned}
$$

(ii)

Note that $\tan\left(\frac{\pi}{4} + x\right) = \frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x}$

$$
\begin{aligned}
BC &= \frac{h}{\tan \frac{\pi}{3}} + \frac{h}{\frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x}} \\
&= \frac{h\sqrt{3}}{3} + \frac{h(1 - \tan x)}{1 + \tan x} \\
&\approx \frac{h\sqrt{3}}{3} + \frac{h(1 - \tan x)}{1 + \tan x} \\
&= \frac{h\sqrt{3}}{3} + h(1 - x)(1 + x)^{-1} \\
&= \frac{h\sqrt{3}}{3} + h(1 - x)[1 + (-1)x + \left(\frac{(-1)(-2)}{2!}\right)x^2 + \ldots] \\
&= \frac{h\sqrt{3}}{3} + h(1 - x)[1 - x + x^2 + \ldots] \\
&= \frac{h\sqrt{3}}{3} + h(1 - 2x + 2x^2 + \ldots) \\
&= h\left(1 + \frac{\sqrt{3}}{3} - 2x + 2x^2\right)
\end{aligned}
$$

6. $\angle QPR = \pi - \left(\frac{\pi}{6} + \theta\right) - \left(\frac{\pi}{6} - \theta\right) = \frac{2\pi}{3}$

By sine rule,

$\frac{PQ}{\sin\left(\frac{\pi}{6} + \theta\right)} = \frac{PR}{\sin\left(\frac{\pi}{6} - \theta\right)} = \frac{3}{\sin \frac{2\pi}{3}} = \frac{3}{\frac{\sqrt{3}}{2}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$

$\Rightarrow PQ = 2\sqrt{3}\sin\left(\frac{\pi}{6} + \theta\right)$ and $PR = 2\sqrt{3}\sin\left(\frac{\pi}{6} - \theta\right)$

Hence,

$$\begin{aligned}
PQ - PR &= 2\sqrt{3}\sin\left(\frac{\pi}{6}+\theta\right) - 2\sqrt{3}\sin\left(\frac{\pi}{6}-\theta\right) \\
&= 2\sqrt{3}\left[\sin\left(\frac{\pi}{6}+\theta\right) - \sin\left(\frac{\pi}{6}-\theta\right)\right] \\
&= 2\sqrt{3}\left[\left(\sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta\right) - \left(\sin\frac{\pi}{6}\cos\theta - \cos\frac{\pi}{6}\sin\theta\right)\right] \\
&\text{OR } 2\sqrt{3}\left\{2\cos\frac{1}{2}\left[\left(\frac{\pi}{6}+\theta\right)+\left(\frac{\pi}{6}-\theta\right)\right]\sin\frac{1}{2}\left[\left(\frac{\pi}{6}+\theta\right)-\left(\frac{\pi}{6}-\theta\right)\right]\right\} \\
&= 4\sqrt{3}\cos\frac{\pi}{6}\sin\theta \\
&= 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\sin\theta \\
&= 6\sin\theta \\
&\approx 6\theta \text{ (since } \theta \text{ is small} \Rightarrow \sin\theta \approx \theta \text{) (shown).} \\
&\therefore a = 2\sqrt{3} \text{ and } b = 6
\end{aligned}$$

7. (a)

Let $f(x) = \frac{x-4}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2}$

Using cover-up rule,

$A = \frac{-1-4}{3(-1)+2} = 5$

$B = \frac{-\frac{2}{3}-4}{-\frac{2}{3}+1} = -14$

$\therefore \frac{x-4}{(x+1)(3x+2)} = \frac{5}{x+1} - \frac{14}{3x+2}$

$$\begin{aligned}
f(x) &= \frac{5}{x-1} - \frac{14}{3x-2} \\
&= 5(1-x)^{-1} - 14(3x-2)^{-1} \\
&= 5(1-x)^{-1} - 14\left[2\left(1+\frac{3x}{2}\right)\right]^{-1} \\
&= 5(1+x)^{-1} - \frac{14}{2}\left(1+\frac{3x}{2}\right)^{-1} \\
&= 5(1-x+x^2-x^3+\ldots) \\
&\quad - 7\left(1+(-1)\left(\frac{3x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3+\ldots\right) \\
&= 5(1-x+x^2-x^3+\ldots) - 7\left(1-\frac{3x}{2}+\frac{9x^2}{4}-\frac{27x^3}{8}+\ldots\right) \\
&= -2+\frac{11}{2}x-\frac{43}{4}x^2+\frac{149}{8}x^3+\ldots
\end{aligned}$$

(b)

Expansion of $(1+x)^{-1}$ is valid for $-1 < x < 1$

Expansion of $\left(1+\frac{3x}{2}\right)^{-1}$ is valid for $-\frac{2}{3} < x < \frac{2}{3}$

Therefore the range of values of x for the expansion of f(x) to be valid is $-\frac{2}{3} < x < \frac{2}{3}$.

(c)

Coefficient of x^n:

$$5(-1)^n - 7(-1)^n\left(\frac{3}{2}\right)^n = (-1)^n\left[5-7\left(\frac{3}{2}\right)^n\right]$$

8. a)

$$\begin{aligned}\left(\frac{1-x}{1+2x}\right)^{\frac{1}{3}} &= (1-x)^{\frac{1}{3}}(1+2x)^{-\frac{1}{3}} \\ &= \left(1-\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(x^2\right)+\ldots\right)\left(1-\frac{2}{3}x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}(2x)^2+\ldots\right) \\ &= \left(1-\frac{1}{3}x-\frac{1}{9}x^2+\ldots\right)\left(1-\frac{2}{3}x+\frac{8}{9}x^2+\ldots\right) \\ &= 1-\frac{2}{3}x+\frac{8}{9}x^2-\frac{1}{3}x+\frac{2}{9}x^2-\frac{1}{9}x^2+\ldots \\ &= 1-x+x^2+\ldots\end{aligned}$$

b) $\frac{1}{6}$ is closer to zero than $-\frac{4}{11}$, hence it gives a better approximation.

c)

$$\begin{aligned}\left(\frac{1-\frac{1}{6}}{1+2\left(\frac{1}{6}\right)}\right)^{\frac{1}{3}} &\approx 1-\left(\frac{1}{6}\right)+\left(\frac{1}{6}\right)^2 \\ \frac{\sqrt[3]{5}}{2} &\approx \frac{31}{36} \\ \sqrt[3]{5} &\approx \frac{31}{18}\end{aligned}$$

