
Complex Numbers

Worked Solutions

2026 A-Level Preparation

21 questions · Sourced from all major JC Preliminary Examinations 2025

- **Section A:** Quadratic equations
- **Section B:** Roots of polynomials (degree 3 & 4)
- **Section C:** Simultaneous equations
- **Section D:** Argand diagram, modulus-argument & loci

How to use these solutions

These worked solutions are intended for *review after attempting the questions independently*. Each solution follows the official marking scheme logic verbatim where possible. Examiner notes highlight the most common errors seen in marking.

●○○ Foundational ●●○ Standard ●●● Challenging

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Q	School	Topic	Marks	Difficulty
Section A — Quadratic equations				
1	RVHS P1	Complex quadratic	6	●●●
Section B — Roots of polynomials (degree 3 & 4)				
2	RVHS P1	Real polynomial, complex root	4	●●○
3	NYJC P2	Polynomials (complex coeff.)	6	●●○
4	SAJC P2	Polynomials (complex coeff.)	9	●●○
5	JPJC P2	Real cubic + Argand	11	●●○
6	VJC P1	Complex equations & cube roots	9	●●●
7	EJC P1	Roots of cubic equation	10	●●●
8	TMJC P1	Real cubic; find roots	11	●●●
9	CJC P2	Complex quartic reasoning	12	●●●
10	HCI P1	Complex polynomial	12	●●●
11	ACJC P1	Real quartic + Argand	13	●●●
Section C — Simultaneous equations				
12	NJC P1	Simultaneous equations	6	●●○
13	SAJC P1	Simultaneous equations	8	●●○
14	TJC P1	Simultaneous equations	10	●●○
Section D — Argand diagram, modulus-argument & loci				
15	RVHS P2	Triangle geometry	5	●●○
16	NJC P2	Modulus-argument form	7	●●○
17	VJC P2	Modulus-argument form	9	●●○
18	NYJC P1	Argand geometry	8	●●●
19	ASRJC P2	Modulus-argument & Argand	10	●●●
20	DHS P1	Complex locus	11	●●●
21	RI P1	Complex analysis	12	●●●

Section A: Quadratic equations

Q1

Q1 RVHS P1 Q3

••• [6 marks]

Find the roots of $z^2 - (1 + 2i)z + 1 + 7i = 0$.

Here $a = 1$, $b = -(1 + 2i)$, $c = 1 + 7i$.

$$\begin{aligned} z &= \frac{(1 + 2i) \pm \sqrt{(-1 - 2i)^2 - 4(1)(1 + 7i)}}{2(1)} \\ &= \frac{1 + 2i \pm \sqrt{4i - 3 - 4 - 28i}}{2} \\ &= \frac{1 + 2i \pm \sqrt{-7 - 24i}}{2} \end{aligned}$$

Find $\sqrt{-7 - 24i}$. Let $\sqrt{-7 - 24i} = x + iy$, $x, y \in \mathbb{R}$.

Then $-7 - 24i = (x + iy)^2 = (x^2 - y^2) + (2xy)i$.

Comparing real and imaginary parts:

$$x^2 - y^2 = -7 \quad \dots (1) \quad 2xy = -24 \implies x = \frac{-12}{y} \quad \dots (2)$$

Substitute (2) into (1):

$$\begin{aligned} \left(\frac{-12}{y}\right)^2 - y^2 = -7 &\implies \frac{144}{y^2} - y^2 = -7 \implies 144 - y^4 = -7y^2 \\ y^4 - 7y^2 - 144 &= 0 \implies (y^2 - 16)(y^2 + 9) = 0 \\ y^2 = 16 \quad \text{or} \quad y^2 = -9 &\text{ (reject, } y \in \mathbb{R}) \implies y = \pm 4 \end{aligned}$$

When $y = 4$: $x = -3$. When $y = -4$: $x = 3$.

So the square roots of $-7 - 24i$ are $3 - 4i$ and $-3 + 4i$.

Hence,

$$\begin{aligned} z &= \frac{1 + 2i \pm (3 - 4i)}{2} \\ z &= \frac{1 + 2i + 3 - 4i}{2} \quad \text{or} \quad z = \frac{1 + 2i - 3 + 4i}{2} \end{aligned}$$

Section B: Roots of polynomials (degree 3 & 4)

Q2–Q11

Q2 RVHS P1 Q1

••• [4 marks]

Method 1 — Using factor theorem and system of equations

Since -1 is a root, $f(-1) = 0$:

$$a(-1)^3 + b(-1)^2 + c(-1) + d = 0 \implies -a + b - c + d = 0 \implies b - c + d - a = 0 \quad \dots (1)$$

Since $4 + i$ is a root, $f(4 + i) = 0$:

$$\begin{aligned} a(4 + i)^3 + b(4 + i)^2 + c(4 + i) + d &= 0 \\ a(52 + 47i) + b(15 + 8i) + c(4 + i) + d &= 0 \end{aligned}$$

Comparing real parts: $52a + 15b + 4c + d = 0 \quad \dots (2)$

Comparing imaginary parts: $47a + 8b + c = 0 \quad \dots (3)$

Solving (1), (2) and (3) using the GC:

Method 2 — Conjugate root theorem

Since all coefficients of $f(x)$ are real, $4 + i$ and $4 - i$ are conjugate roots of $f(x) = 0$.

$$\begin{aligned} f(x) &= a[x - (4 + i)][x - (4 - i)][x + 1] \\ &= a[x^2 - 8x + 17][x + 1] \\ &= a[x^3 - 7x^2 + 9x + 17] \\ &= ax^3 - 7ax^2 + 9ax + 17a \end{aligned}$$

Q3 NYJC P2 Q2

••• [6 marks]

(a)

Replacing $a = \lambda i$ and $b = \mu i$ where $\lambda, \mu \in \mathbb{R}$:

$$-ix^3 + 5ix^2 + \lambda ix + \mu i = 0 \implies x^3 - 5x^2 - \lambda x - \mu = 0.$$

Since the coefficients of this equation are all real, the complex roots occur in conjugate pairs.

(b)

The equation $ix^3 + 5ix^2 + a^*x + b = 0$.

Since a is purely imaginary, $a^* = -a$. Using the substitution $x \rightarrow -x$:

$$-i(-x)^3 + 5i(-x)^2 + a(-x) + b = 0.$$

This is exactly the equation in **(a)**, which has roots $x = 2 + i, 2 - i, 1$.

So the substitution gives $-x = 2 + i, 2 - i, 1$, yielding:

Q4 SAJC P2 Q2

••• [9 marks]

(a)(i) Since $x = \alpha$ is a root, $f(\alpha) = 0$:

$$p\alpha^6 + q\alpha^4 + r = 0$$

$$f(-\alpha) = p(-\alpha)^6 + q(-\alpha)^4 + r = p\alpha^6 + q\alpha^4 + r = 0$$

(a)(ii) $x = 5$ and $x = \beta$ are roots. By part (i), $x = -5$ and $x = -\beta$ are also roots. Since the coefficients of f are all real, $x = \beta^*$ and $x = -\beta^*$ are also roots.

(b)

Method 1

Since $1 + i$ is a root:

$$z^3 + (3 - a)z^2 - (2 + 6i)z - 6 = (z - 1 - i)(z^2 + Bz + C)$$

Comparing constant terms:

$$-6 = (-1 - i)C \implies C = \frac{-6}{-1 - i} = \frac{6}{1 + i} = \frac{6(1 - i)}{2} = 3 - 3i$$

Comparing z coefficients:

$$-(2 + 6i) = B(-1 - i) + C = B(-1 - i) + (3 - 3i)$$

$$B(-1 - i) = -(2 + 6i) - (3 - 3i) = -5 - 3i$$

$$B = \frac{-5 - 3i}{-1 - i} = \frac{(5 + 3i)(1 - i)}{(1 + i)(1 - i)} = \frac{5 - 5i + 3i + 3}{2} = \frac{8 - 2i}{2} = 4 - i$$

Comparing z^2 coefficients:

$$3 - a = -1 - i + B = -1 - i + 4 - i = 3 - 2i \implies a = 2i$$

So:

$$z^3 + (3 - 2i)z^2 - (2 + 6i)z - 6 = (z - 1 - i)(z^2 + (4 - i)z + 3 - 3i) = 0$$

Solving $z^2 + (4 - i)z + 3 - 3i = 0$:

$$(z + 3)(z + 1 - i) = 0 \implies z = -3 \quad \text{or} \quad z = -1 + i$$

OR using the quadratic formula:

$$z = \frac{-(4 - i) \pm \sqrt{(4 - i)^2 - 4(3 - 3i)}}{2} = \frac{-(4 - i) \pm \sqrt{15 - 8i - 12 + 12i}}{2} = \frac{-(4 - i) \pm \sqrt{3 + 4i}}{2}$$

$$\sqrt{3 + 4i} = 2 + i \implies z = \frac{-4 + i \pm (2 + i)}{2} \implies z = -3 \quad \text{or} \quad z = -1 + i$$

Q5 JPJC P2 Q4

••• [11 marks]

- (a) **Method 1** Since the polynomial has real coefficients, if $3 - i$ is a root then so is $3 + i$.

$$\begin{aligned} [w - (3 - i)][w - (3 + i)] &= [(w - 3) + i][(w - 3) - i] \\ &= (w - 3)^2 - i^2 \\ &= w^2 - 6w + 10. \end{aligned}$$

Write $5w^3 + pw^2 + 68w + q = (w^2 - 6w + 10)(5w + a)$. Expanding:

$$(w^2 - 6w + 10)(5w + a) = 5w^3 + aw^2 - 30w^2 - 6aw + 50w + 10a.$$

Comparing coefficients:

$$w^2 : a - 30 = p \quad (1)$$

$$w : -6a + 50 = 68 \quad (2)$$

$$\text{const} : 10a = q \quad (3)$$

From (2): $a = -3$. From (1): $p = -3 - 30 = -33$. From (3): $q = -30$.

So $5w^3 - 33w^2 + 68w - 30 = (w^2 - 6w + 10)(5w - 3) = 0$.

The other roots are $3 + i$ and $\frac{3}{5}$.

Method 2 (substitute the root) With $w = 3 - i$, $w^2 = 8 - 6i$, $w^3 = 18 - 26i$. Substituting into $5w^3 + pw^2 + 68w + q = 0$:

$$5(18 - 26i) + p(8 - 6i) + 68(3 - i) + q = 0,$$

i.e. $90 - 130i + 8p - 6pi + 204 - 68i + q = 0$.

Comparing real and imaginary parts:

$$\text{Im} : -130 - 6p - 68 = 0 \Rightarrow p = -33,$$

$$\text{Re} : 90 + 8(-33) + 204 + q = 0 \Rightarrow q = -30.$$

Then factorising as in Method 1 gives the other roots $3 + i$ and $\frac{3}{5}$.

Q6 VJC P1 Q7

●●● [9 marks]

(a)

Let $z = a + bi$, where $a, b \in \mathbb{R}$.

$$\begin{aligned}\frac{4|z|}{15 - z^*} &= 5i \\ \frac{4\sqrt{a^2 + b^2}}{15 - (a - bi)} &= 5i \\ 4\sqrt{a^2 + b^2} &= 75i - 5ai - 5b = -5b + (75 - 5a)i\end{aligned}$$

Comparing real and imaginary parts:

$$\begin{cases} 4\sqrt{a^2 + b^2} = -5b & \text{--- (1)} \\ 75 - 5a = 0 \implies a = 15 & \text{--- (2)} \end{cases}$$

Substituting (2) into (1):

$$\begin{aligned}4\sqrt{15^2 + b^2} &= -5b \\ 4^2(15^2 + b^2) &= 25b^2 \\ 9b^2 &= 3600 \\ b^2 &= 400 \\ b &= -20 \quad (\because b < 0 \text{ by eqn. (1)})\end{aligned}$$

$$\therefore z = 15 - 20i$$

(b)(i)

$$\begin{aligned}(w - i)^3 &= -i \\ w^3 + 3w^2(-i) + 3w(-i)^2 + (-i)^3 + i &= 0 \\ w^3 - 3iw^2 - 3w + 2i &= 0\end{aligned}$$

Since $2i$ is a root,

$$w^3 - 3iw^2 - 3w + 2i = (w - 2i)(Aw^2 + Bw + C)$$

By comparing coefficients, $A = 1$, $C = -1$.

$$C - 2iB = -3 \implies B = \frac{-1 + 3}{2i} = -i$$

$$w^2 - iw - 1 = 0$$

$$w = \frac{i \pm \sqrt{(-i)^2 - 4(-1)}}{2}$$

$$w = \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{or} \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

(b)(ii)

Let W_1 , W_2 , W_3 and A represent $2i$, $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and ki respectively.

For $W_1A = W_2A$, i.e. $\left| \frac{\sqrt{3}+i}{2} - ki \right| = |2i - ki|$,

$$\begin{aligned} \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} - k\right)^2 &= (2 - k)^2 \\ \frac{3}{4} + 1 - k + k^2 &= 4 - 4k + k^2 \\ k &= 1 \end{aligned}$$

So, we have $W_1A = |2i - i| = 1$ and $W_2A = \left| \frac{\sqrt{3}+i}{2} - i \right| = 1$.

Check that $W_3A = 1$ too:

$$\left| \frac{-\sqrt{3}+i}{2} - i \right| = \left| \frac{-\sqrt{3}-i}{2} \right| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = 1$$

Since the distances are all equal to 1, W_1 , W_2 and W_3 lie on a circle with radius 1, centred at A representing the complex number i .

Q7 EJC P1 Q8

••• [10 marks]

$$\begin{aligned}
 (a + bi)^2 &= 2i \\
 a^2 - b^2 + 2abi &= 2i \\
 a^2 - b^2 &= 0 \quad \text{--- (1)} \\
 2ab &= 2 \quad \text{--- (2)}
 \end{aligned}$$

From (2), $a = \frac{1}{b}$

$$\begin{aligned}
 \left(\frac{1}{b}\right)^2 - b^2 &= 0 \\
 1 - b^4 &= 0 \\
 (1 + b^2)(1 - b^2) &= 0 \\
 b^2 &= 1 \quad (\because b \text{ is real})
 \end{aligned}$$

So $b = 1$, $a = 1$ or $b = -1$, $a = -1$ (reject since $0 \leq \arg(w) \leq \frac{\pi}{2}$)

$$w = 1 + i$$

Write:

$$z^3 - z^2 + (1 - i)z + s = [z - (1 + i)](z^2 + Bz + C)$$

Expanding and comparing coefficients:

$$\begin{array}{ll}
 z^2 : & -1 = -(1 + i) + B & \implies B = i \\
 z : & 1 - i = -(1 + i)B + C = -(1 + i)i + C = 1 - i + C & \implies C = 0 \\
 \text{const} : & s = -(1 + i)C = 0 & \implies s = 0
 \end{array}$$

So $z^2 + iz = 0$, giving $z(z + i) = 0$.

Other roots: $z = 0$ and $z = -i$. $s = 0$.

Replace z with iv : $(iv)^3 - (iv)^2 + (1 - i)(iv) + s = 0$

$$-iv^3 + v^2 + (1 + i)v + s = 0$$

Hence, $z = iv \implies v = -iz$

$$\begin{aligned}
 v &= -i(1 + i) = 1 - i, \\
 v &= -i(0) = 0 \quad \text{or} \\
 v &= -i(-i) = -1
 \end{aligned}$$

Q8 TMJC P1 Q9

... [11 marks]

(a) Since the equation has real coefficients, $-2+2i$ being a root implies $-2-2i$ is also a root. Consider

$$\begin{aligned} [z - (-2 + 2i)][z - (-2 - 2i)] &= [(z + 2) - 2i][(z + 2) + 2i] \\ &= (z + 2)^2 - (2i)^2 \\ &= z^2 + 4z + 4 + 4 \\ &= z^2 + 4z + 8. \end{aligned}$$

Writing the cubic as $(z^2 + 4z + 8)(pz + q)$:

- Coefficient of z^3 : $p = 1$.
- Constant: $8q = -16\sqrt{2} \Rightarrow q = -2\sqrt{2}$.

So $z^3 + az^2 + bz - 16\sqrt{2} = (z^2 + 4z + 8)(z - 2\sqrt{2})$.

- Coefficient of z^2 : $a = -2\sqrt{2} + 4 = 4 - 2\sqrt{2}$.
- Coefficient of z : $b = -8\sqrt{2} + 8 = 8 - 8\sqrt{2}$.

Therefore $a = 4 - 2\sqrt{2}$, $b = 8 - 8\sqrt{2}$, and the other two roots are $-2 - 2i$ and $2\sqrt{2}$.

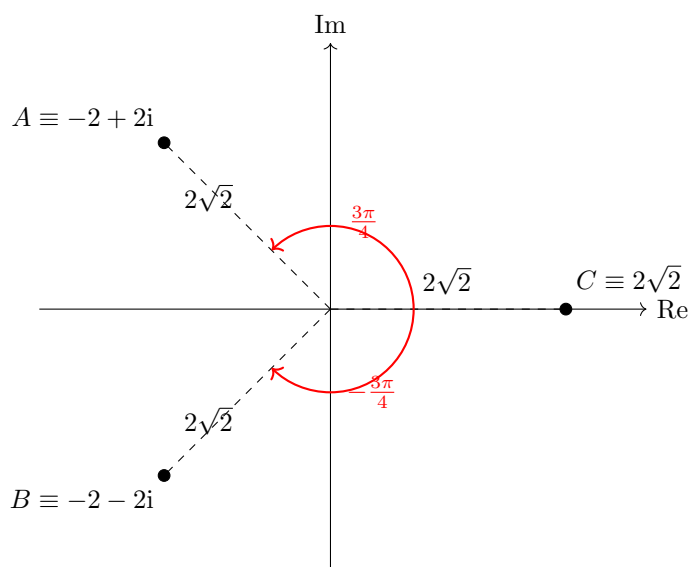
(b) Let $z_1 = -2 + 2i$ (2nd quadrant):

$$\begin{aligned} |z_1| &= \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}, \\ \arg(z_1) &= \pi - \tan^{-1}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}. \end{aligned}$$

Let $z_2 = -2 - 2i$ (3rd quadrant):

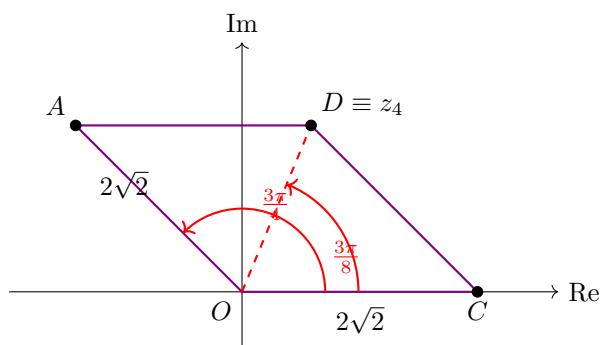
$$\begin{aligned} |z_2| &= 2\sqrt{2}, \\ \arg(z_2) &= -\pi + \tan^{-1}\left(\frac{2}{2}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}. \end{aligned}$$

Let $z_3 = 2\sqrt{2}$: $|z_3| = 2\sqrt{2}$ and $\arg(z_3) = 0$.



Geometrical relationship: B is the reflection of A in the real axis (equivalently, B is the 90° anti-clockwise rotation of A about O , since $iA = i(-2 + 2i) = -2 - 2i = B$).

(c) Since $OA = OC = 2\sqrt{2}$, consider the point D such that $OADC$ is a rhombus. Then OD bisects $\angle AOC$.



Thus $\angle DOC = \frac{1}{2} \cdot \frac{3\pi}{4} = \frac{3\pi}{8}$.

Let $D \equiv z_4$ where

$$z_4 = (-2 + 2i) + 2\sqrt{2} = (-2 + 2\sqrt{2}) + 2i.$$

Since z_4 lies in the 1st quadrant, $\arg(z_4) = \frac{3\pi}{8}$:

$$\begin{aligned} \tan \frac{3\pi}{8} &= \frac{2}{-2 + 2\sqrt{2}} \\ &= \frac{1}{-1 + \sqrt{2}} \times \frac{-1 - \sqrt{2}}{-1 - \sqrt{2}} \\ &= \frac{-1 - \sqrt{2}}{(-1)^2 - (\sqrt{2})^2} \\ &= \frac{-1 - \sqrt{2}}{1 - 2} \\ &= 1 + \sqrt{2} \quad (\text{shown}). \end{aligned}$$

Q9 CJC P2 Q4

●●● [12 marks]

(a) The student's claim may not be true as p may not be real.

(b) Substituting $z = 3 + i$:

$$\begin{aligned}
 2z^4 - 14z^3 + 33z^2 - 26z + p &= 0 \\
 2(3 + i)^4 - 14(3 + i)^3 + 33(3 + i)^2 - 26(3 + i) + p &= 0 \\
 2(8 + 6i)(8 + 6i) - 14(3 + i)(8 + 6i) + 33(8 + 6i) - 78 - 26i + p &= 0 \\
 2(28 + 96i) - 14(18 + 26i) + 264 + 198i - 78 - 26i + p &= 0 \\
 56 + 192i - 252 - 364i + 264 + 198i - 78 - 26i + p &= 0 \\
 -10 + p &= 0 \\
 p &= 10
 \end{aligned}$$

(c) Since all coefficients are real and $3 + i$ is a root, $3 - i$ is also a root.

$$[z - (3 + i)][z - (3 - i)] = (z - 3)^2 + 1 = z^2 - 6z + 10$$

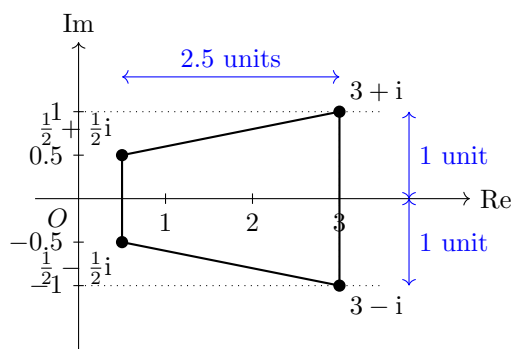
Dividing $2z^4 - 14z^3 + 33z^2 - 26z + 10$ by $z^2 - 6z + 10$:

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = (z^2 - 6z + 10)(2z^2 - 2z + 1)$$

Solving $2z^2 - 2z + 1 = 0$:

$$z = \frac{2 \pm \sqrt{4 - 8}}{4} = \frac{2 \pm \sqrt{-4}}{4} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$$

The four roots are: $3 + i$, $3 - i$, $\frac{1}{2} + \frac{1}{2}i$, $\frac{1}{2} - \frac{1}{2}i$.



(d) The quadrilateral is a **trapezium**.

The two pairs of opposite sides: the vertical side joining $3 + i$ and $3 - i$ has length 2; the vertical side joining $\frac{1}{2} + \frac{1}{2}i$ and $\frac{1}{2} - \frac{1}{2}i$ has length 1. Both are vertical (parallel). The horizontal width is $3 - \frac{1}{2} = 2.5$.

$$\text{Area} = \frac{1}{2}(1 + 2)(2.5) = 3.75 \text{ units}^2 \quad (= \frac{15}{4} \text{ units}^2)$$

Q10 HCI P1 Q12

●●● [12 marks]

(a) *Assume* (for contradiction) that $f(x) = 0$ has a real root $x \in \mathbb{R}$. Then $x^4 - 6x^2 + k = 0$, i.e. $k = -x^4 + 6x^2$. The right-hand side is real, but k is purely imaginary (and non-zero), so $k \notin \mathbb{R}$ contradiction. Hence $f(z) = 0$ **cannot have real roots** when k is purely imaginary.

(b) Direct substitution:

$$\begin{aligned} f(-z) &= (-z)^4 - 6(-z)^2 + k \\ &= z^4 - 6z^2 + k \\ &= f(z). \quad \text{(Shown)} \end{aligned}$$

(c)

$$\begin{aligned} (2+i)^2 &= 2^2 + 2(2)(i) + i^2 = 3 + 4i \\ (2+i)^4 &= [(2+i)^2]^2 \\ &= (3+4i)^2 \\ &= 3^2 + 2(3)(4i) + (4i)^2 \\ &= -7 + 24i \end{aligned}$$

$$\begin{aligned} (2+i)^4 - 6(2+i)^2 + k &= 0 \\ -7 + 24i - 6(3+4i) + k &= 0 \\ k &= 25 \end{aligned}$$

Since all coefficients of $f(z) = 0$ are real and $2+i$ is a root, then $2-i$ is also a root.

Since $f(-z) = f(z)$, then $-z$ is also a root of $f(z) = 0$.

The remaining roots are $2-i$, $-2-i$ and $-2+i$.

(d)

Method 1

$$\begin{aligned} (2+i)(2-i) &= |2+i|^2 = 5 \\ (-2+i)(-2-i) &= |-2+i|^2 = 5 \\ \therefore D &= 5(5) = 25 \end{aligned}$$

Method 2

$$\begin{aligned} (2+i)(2-i) &= 2^2 - i^2 = 5 \\ (-2+i)(-2-i) &= (-2)^2 - i^2 = 5 \\ \therefore D &= 5(5) = 25 \end{aligned}$$

(e) $z^4 - 6z^2 + k = 0$, so $\frac{k}{z^4} - \frac{6}{z^2} + 1 = 0$.

Replace $\frac{1}{z}$ with w , i.e. $w = \frac{1}{z}$:

$$\begin{aligned}w_1 &= \frac{1}{2+i} \\ &= \frac{2-i}{2-i} \\ &= \frac{2-i}{2-i^2} \\ &= \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i\end{aligned}$$

Alternative

$$\begin{aligned}25w^4 - 6w^2 + 1 &= 0 \\ 25^2w^4 - 6(25)w^2 + 25 &= 0 \\ (5w)^4 - 6(5w)^2 + 25 &= 0 \\ 5w &= 2 + i \\ w &= \frac{2}{5} + \frac{1}{5}i\end{aligned}$$

Q11 ACJC P1 Q10

●●● [13 marks]

(a)

Since the coefficients are all real, complex roots occur in conjugate pairs. Therefore $2 - 3i$ is also a root.

$$\omega^4 - 2\omega^3 + 10\omega^2 + p\omega + q = (\omega - (2 + 3i))(\omega - (2 - 3i))(\omega^2 + a\omega + b) = (\omega^2 - 4\omega + 13)(\omega^2 + a\omega + b)$$

Comparing coefficients:

$$\omega^3 : -2 = -4 + a \implies a = 2$$

$$\omega^2 : 10 = -4a + 13 + b = 5 + b \implies b = 5$$

$$\text{So } \omega^4 - 2\omega^3 + 10\omega^2 + p\omega + q = (\omega^2 - 4\omega + 13)(\omega^2 + 2\omega + 5).$$

$$\omega^1 : p = -20 + 26 = 6$$

$$\omega^0 : q = 13 \times 5 = 65$$

The other two roots from $\omega^2 + 2\omega + 5 = 0$:

$$\omega = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

$$p = 6, \quad q = 65; \quad \text{other roots: } 2 - 3i, \quad -1 + 2i, \quad -1 - 2i.$$

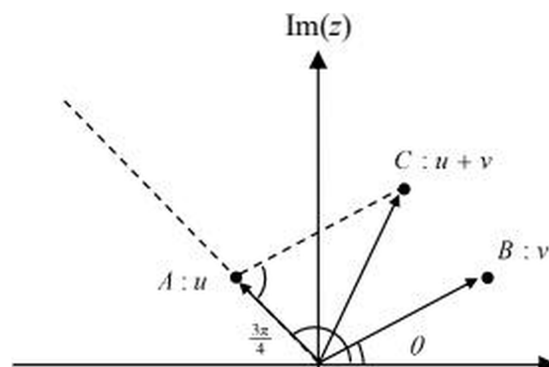
(b(i))

$$|u| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2 + 2} = 2$$

$$\text{Basic angle} = \tan^{-1} \frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi}{4}. \quad u \text{ is in the 2nd quadrant, so } \arg u = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(b(ii))

A is at distance 2 from the origin at argument $\frac{3\pi}{4}$ (second quadrant). B is at distance 3 from the origin at argument θ (first quadrant, $0 < \theta < \frac{\pi}{4}$). C represents $u + v$, forming a parallelogram $OACB$.



(b(iii))

$$\angle AOB = \frac{3\pi}{4} - \theta, \quad \angle OAC = \pi - \angle AOB = \pi - \frac{3\pi}{4} + \theta = \theta + \frac{\pi}{4}.$$

Applying the cosine rule on $\triangle OAC$ ($OA = |u| = 2$, $AC = |v| = 3$, $OC = |u + v|$):

$$\begin{aligned} |u + v|^2 &= 2^2 + 3^2 - 2(2)(3) \cos\left(\theta + \frac{\pi}{4}\right) \\ &= 13 - 12 \cos\left(\theta + \frac{\pi}{4}\right) \end{aligned}$$

Section C: Simultaneous equations

Q12–Q14

Q12 NJC P1 Q2

••• [6 marks]

(i)

$$(1 + i)z + 2w = -2 + 4i \quad \dots (1)$$

$$3z - w = 4 + 2i \quad \dots (2)$$

$$\text{Multiply (2) by 2: } 6z - 2w = 8 + 4i \quad \dots (3)$$

Adding (1) and (3):

$$\begin{aligned} (6 + 1 + i)z &= -2 + 4i + 8 + 4i \\ (7 + i)z &= 6 + 8i \\ z &= \frac{6 + 8i}{7 + i} \times \frac{7 - i}{7 - i} \\ &= \frac{42 - 6i + 56i + 8}{50} \\ &= \frac{50 + 50i}{50} \\ &= 1 + i \end{aligned}$$

From (2):

$$\begin{aligned} w &= 3z - 4 - 2i \\ &= 3(1 + i) - 4 - 2i \\ &= -1 + i \end{aligned}$$

$$z = 1 + i, \quad w = -1 + i$$

(ii)

$$\begin{aligned} \frac{w}{z} &= \frac{-1 + i}{1 + i} \times \frac{1 - i}{1 - i} \\ &= \frac{-1 + i + i + 1}{1^2 + 1^2} \\ &= \frac{2i}{2} = i \end{aligned}$$

Since $w = iz$, multiplying by i corresponds to a rotation of $\frac{\pi}{2}$ anti-clockwise about the origin.

Q13 SAJC P1 Q10

●●● [8 marks]

(a)

$$|z| + 5w = 0 \quad (1)$$

$$iz - 4w = -4 + 7i \quad (2)$$

From (1): $w = -\frac{1}{5}|z|$ (3)

Substitute (3) into (2):

$$iz + \frac{4}{5}|z| = -4 + 7i$$

Let $z = x + yi$:

$$i(x + yi) + \frac{4}{5}\sqrt{x^2 + y^2} = -4 + 7i$$

$$\left(-y + \frac{4}{5}\sqrt{x^2 + y^2}\right) + xi = -4 + 7i$$

Comparing imaginary parts: $x = 7$.

Comparing real parts:

$$-y + \frac{4}{5}\sqrt{49 + y^2} = -4$$

$$\frac{4}{5}\sqrt{49 + y^2} = y - 4 \implies 5(y - 4) = 4\sqrt{49 + y^2}$$

Squaring (noting $y \geq 4$ required):

$$25(y - 4)^2 = 16(49 + y^2)$$

$$25(16 - 8y + y^2) = 16(49 + y^2)$$

$$25y^2 - 200y + 400 = 784 + 16y^2$$

$$9y^2 - 200y - 384 = 0 \implies (y - 24)(9y + 16) = 0$$

$$y = 24 \quad \text{or} \quad y = -\frac{16}{9}$$

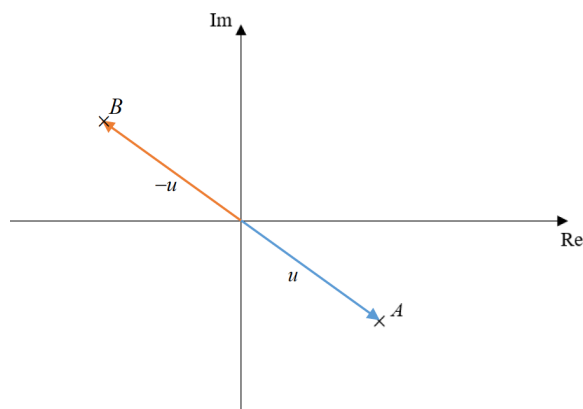
Note that $y = -\frac{16}{9}$ does not satisfy (*) (since $y \geq 4$ required), hence $y = 24$.

$$\therefore z = 7 + 24i$$

Substituting into (3):

$$w = -\frac{1}{5}\sqrt{7^2 + 24^2} = -\frac{1}{5}\sqrt{625} = -5$$

(b)(i)



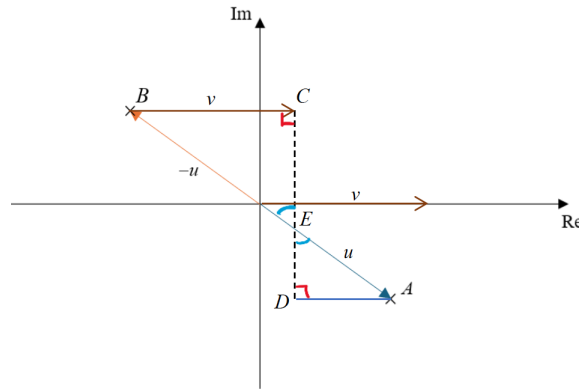
(b)(ii) As C and D represent a complex conjugate pair, line segment CD is **vertical**, and C, D are equidistant from the real axis.

Given $\angle CDA = 90$ and CD is vertical, line segment DA must be **horizontal**.

Since A and B are equidistant from the real axis, DA horizontal implies BC is also horizontal.

Since C represents $v - u$, the vector \overrightarrow{BC} represents v .

Since \overrightarrow{BC} is horizontal, v must be real.



Q14 TJC P1 Q8

••• [10 marks]

(a)

$$2z + |w| = 2 - 4i \quad \dots (1)$$

$$iz - w = 2i \quad \dots (2)$$

From (1): $z = \frac{2 - 4i - |w|}{2}$.

Substitute into (2):

$$i\left(\frac{2 - 4i - |w|}{2}\right) - w = 2i \implies 2i - 4i^2 - i|w| - 2w = 4i \implies 4 + 2i - i|w| - 2w = 4i$$

Let $w = a + bi$, $a, b \in \mathbb{R}$, so $|w| = \sqrt{a^2 + b^2}$:

$$4 + 2i - i\sqrt{a^2 + b^2} - 2(a + bi) = 4i$$

$$(4 - 2a) + i(2 - \sqrt{a^2 + b^2} - 2b) = 4i$$

Comparing real parts: $4 - 2a = 0 \implies a = 2 \quad \dots (3)$ Comparing imaginary parts: $2 - \sqrt{4 + b^2} - 2b = 4 \implies -2 - 2b = \sqrt{4 + b^2} \quad \dots (4), (5)$ (Requires $-2 - 2b \geq 0$, i.e. $b \leq -1$.)

Squaring:

$$(2 + 2b)^2 = 4 + b^2 \implies 4 + 8b + 4b^2 = 4 + b^2 \implies 3b^2 + 8b = b(3b + 8) = 0$$

$$b = 0 \quad (\text{rejected since } \sqrt{4 + b^2} \neq -2 \text{ from (5)}) \quad \text{or} \quad b = -\frac{8}{3}$$

$$z = \frac{2 - 4i - (-(-2 - 2b))}{2} = \frac{2 - 4i - \frac{10}{3}}{2} = \frac{-\frac{4}{3} - 4i}{2} = -\frac{2}{3} - 2i$$

(b) Argand diagram with $A(2, -1)$ and $C(-3, 2)$. \overrightarrow{BA} represents $2 - i - z_2$ and \overrightarrow{BC} represents $-3 + 2i - z_2$.Since $\angle ABC = \frac{\pi}{2}$, A is obtained by rotating C through 90° anticlockwise about B :

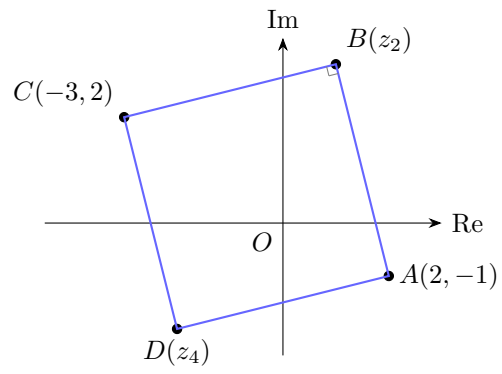
$$i(-3 + 2i - z_2) = 2 - i - z_2$$

$$-3i - 2 - iz_2 = 2 - i - z_2 \implies (1 - i)z_2 = 4 + 2i$$

$$z_2 = \frac{4 + 2i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{4 + 4i + 2i - 2}{2} = \frac{2 + 6i}{2} = 1 + 3i$$

Since $ABCD$ is a parallelogram, $\overrightarrow{AB} = \overrightarrow{DC}$:

$$1 + 3i - (2 - i) = -3 + 2i - z_4 \implies -1 + 4i = -3 + 2i - z_4 \implies z_4 = -2 - 2i$$



Section D: Argand diagram, modulus-argument & loci

Q15–Q21

Q15 RVHS P2 Q2

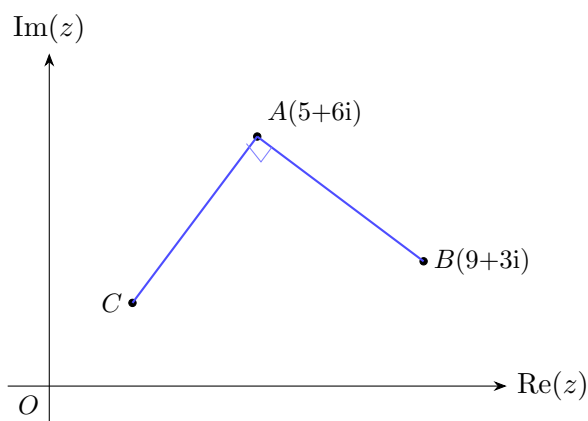
••• [5 marks]

(a) The line segment AC can be obtained by rotating the line segment AB , 90° in a clockwise direction about A .

$$z_C - z_A = -i(z_B - z_A)$$

$$z_C = z_A - i(z_B - z_A)$$

$$z_C = (5 + 6i) - i[(9 + 3i) - (5 + 6i)] = (5 + 6i) - i[4 - 3i] = 2 + 2i$$



(b)

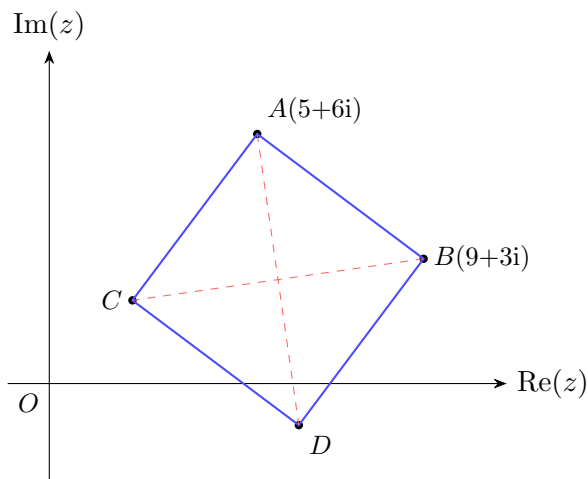
Method 1 — Midpoints of diagonals of parallelogram $ABDC$

$ABDC$ is a parallelogram, so midpoint of AD = midpoint of BC :

$$\frac{z_A + z_D}{2} = \frac{z_B + z_C}{2}$$

$$(5 + 6i) + z_D = (9 + 3i) + (2 + 2i)$$

$$z_D = 11 + 5i - (5 + 6i)$$



Q16 NJC P2 Q1

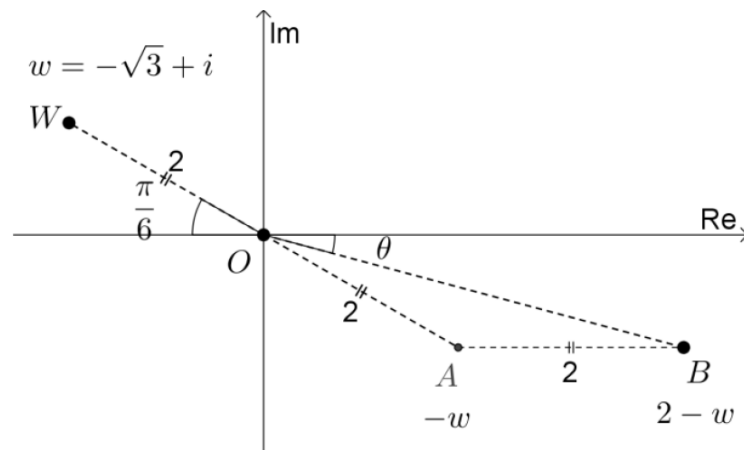
••• [7 marks]

(i)

$$|w| = |i - \sqrt{3}| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$$

$$\arg(w) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

(ii)



$|w| = |-w| = |2-w| = 2$, so O , A (representing $-w$) and B (representing $2-w$) all lie on a circle of radius 2 centred at the origin. Since AB is parallel to the real axis, triangle OAB is isosceles.

(iii)

Since $|w| = 2$, triangle OAB is isosceles with AB parallel to the real axis, so $\angle AOB = \angle ABO = \theta$.

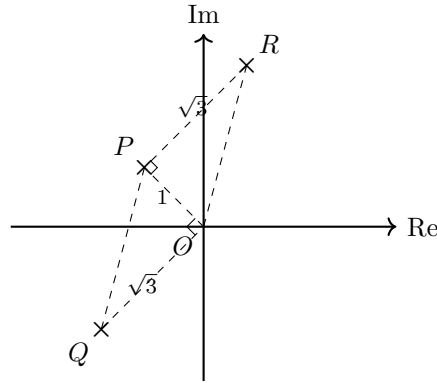
$$2\theta = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{12}$$

Q17 VJC P2 Q4

••• [9 marks]

(a) Since $w = i\sqrt{3}z$, we have $|w| = \sqrt{3}$ and $\arg w = \arg z + \frac{\pi}{2}$, so $OQ \perp OP$ at O with $|OP| = 1$ and $|OQ| = \sqrt{3}$. Also $z - w = z(1 - i\sqrt{3})$ with $|1 - i\sqrt{3}| = 2$, so $|OR| = 2$; and $\vec{PR} = -\vec{OQ}$, so $PR \perp OP$ at P with $|PR| = \sqrt{3}$. Hence R and Q lie on opposite sides of line OP , each at perpendicular distance $\sqrt{3}$ from OP .



(b)

$$\text{Area}(ORPQ) = \frac{1}{2}(1)(\sqrt{3} + \sqrt{3}) = \sqrt{3}.$$

(c)

$$\text{Length } x = 1 \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\text{Length } y = 1 \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

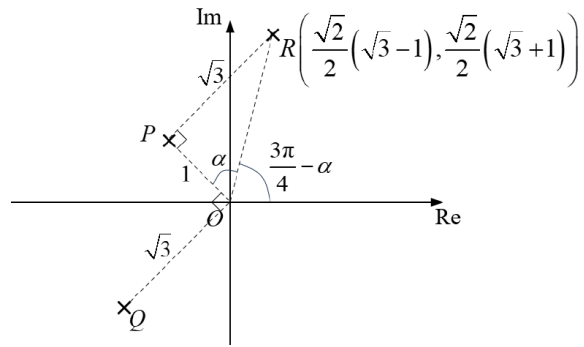
$$\therefore z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

(d)

$$\begin{aligned} z - w &= z(1 - i\sqrt{3}) = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)(1 - i\sqrt{3}) \\ &= \frac{1}{\sqrt{2}}(-1 + i)(1 - i\sqrt{3}) \\ &= \frac{1}{\sqrt{2}}(-1 + i\sqrt{3} + i + \sqrt{3}) \\ &= \frac{1}{\sqrt{2}}[(\sqrt{3} - 1) + (\sqrt{3} + 1)i] \end{aligned}$$

$$\text{Hence } k = \frac{1}{\sqrt{2}}.$$

(e)



$$\alpha = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$\tan\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \frac{\frac{\sqrt{2}}{2}(\sqrt{3}+1)}{\frac{\sqrt{2}}{2}(\sqrt{3}-1)}$$

$$\tan \frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

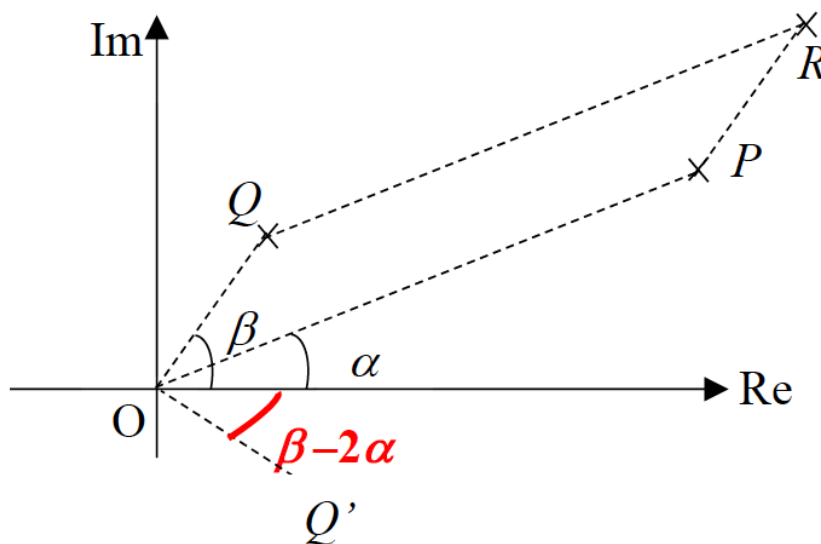
Q18 NYJC P1 Q6

●●● [8 marks]

(a) If $|p| = |q|$, then $OPRQ$ forms a **rhombus**.

$$\arg(r) = \frac{1}{2}(\beta - \alpha) + \alpha = \frac{1}{2}(\alpha + \beta)$$

(b)



$$\angle POQ' = \beta - \alpha$$

(b)(i)

Method 1: $\arg(q') = -(\beta - 2\alpha) = 2\alpha - \beta$ (since $\beta > 2\alpha$, q' is below the real axis)

Method 2: $\arg(q') = \alpha - (\beta - \alpha) = 2\alpha - \beta$

(b)(ii) $|q'| = |q|$. Let F be the foot of perpendicular from Q' to the real axis.

Real part of $q' = |q| \cos(-\beta + 2\alpha)$ (adjacent side of $\triangle OQ'F$)

Imaginary part of $q' = |q| \sin(-\beta + 2\alpha)$ (opposite side of $\triangle OQ'F$, and negative)

$$\therefore q' = |q| \cos(\beta - 2\alpha) - i|q| \sin(\beta - 2\alpha)$$

Q19 ASRJJC P2 Q6

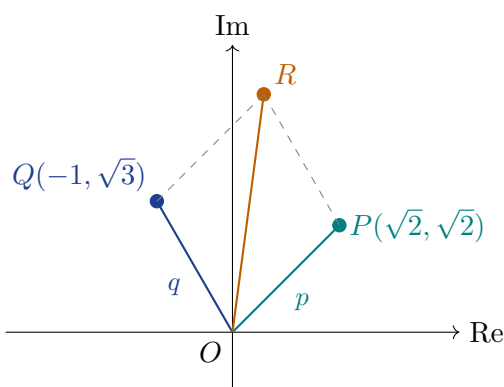
●●● [10 marks]

(a)

$$|p| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2,$$

$$\arg(p) = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

$$|p| = 2, \quad \arg(p) = \frac{\pi}{4}.$$

(b) Argand diagram with $P = \sqrt{2} + \sqrt{2}i$, $|q| = 2$, $\arg(q) = 2\pi/3$, and $R = p + q$:

(c)

Let F be the foot of perpendicular from Q to the y -axis.

$$\operatorname{Re}(q) = -QF = -|q| \sin \frac{2\pi}{3} = -1$$

$$\operatorname{Im}(q) = OF = |q| \cos \frac{2\pi}{3} = \sqrt{3}$$

$$\operatorname{Re}(q) = -1, \quad \operatorname{Im}(q) = \sqrt{3}.$$

(d) Since $|p| = |q| = 2$, $OPRQ$ is a parallelogram with equal adjacent sides. Quadrilateral $OPRQ$ is a **rhombus**.(e) Since $OPRQ$ is a rhombus, the diagonal OR bisects $\angle POQ$. Therefore:

$$\begin{aligned} \arg(p+q) &= \frac{1}{2}(\arg(p) + \arg(q)) \\ &= \frac{1}{2}\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) \\ &= \frac{1}{2} \cdot \frac{11\pi}{12} \\ &= \frac{11\pi}{24}. \end{aligned}$$

Also, $p + q = (\sqrt{2} + \sqrt{2}i) + (-1 + \sqrt{3}i) = (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{3})i$.

Therefore:

$$\tan \frac{11\pi}{24} = \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - 1}.$$

Rationalise by multiplying numerator and denominator by $(\sqrt{2} + 1)$:

$$\begin{aligned}\tan \frac{11\pi}{24} &= \frac{(\sqrt{2} + \sqrt{3})(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \\ &= \frac{2 + \sqrt{2} + \sqrt{6} + \sqrt{3}}{2 - 1} \\ &= \sqrt{6} + \sqrt{3} + \sqrt{2} + 2. \quad (\text{Shown})\end{aligned}$$

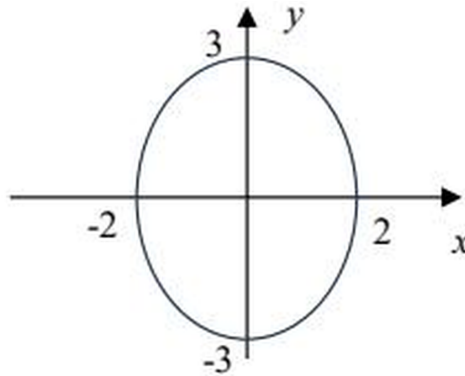
Q20 DHS P1 Q8

●●● [11 marks]

(a) With $x = \operatorname{Re}(z) = 2 \cos t$ and $y = \operatorname{Im}(z) = 3 \sin t$:

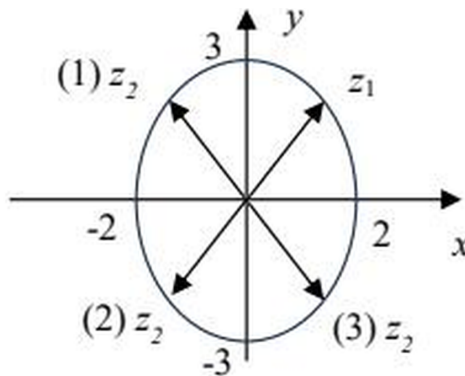
$$\cos t = \frac{x}{2}, \quad \sin t = \frac{y}{3} \implies \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Cartesian equation: $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (an ellipse with semi-axes $a = 2$, $b = 3$)



(b) Since $|z_1| = |z_2|$, both points are equidistant from the origin on the ellipse. With z_1 in the first quadrant (represented as (a, b) , $a, b > 0$), there are three possible positions for z_2 :

- (1) $z_2 = (-a, b)$: then $z_1 + z_2 = (0, 2b)$, so $\arg(z_1 + z_2) = \frac{\pi}{2}$
- (2) $z_2 = (-a, -b)$: then $z_1 + z_2 = 0$, so $\arg(z_1 + z_2)$ is **undefined**
- (3) $z_2 = (a, -b)$: then $z_1 + z_2 = (2a, 0)$, so $\arg(z_1 + z_2) = 0$



$\arg(z_1 + z_2)$ is either undefined, 0, or $\frac{\pi}{2}$.

(c)

From the diagram, if z_2 is at position (1), α will not be real.

$$(z - z_1)(z - z_2) = 0 \implies z^2 - z(z_1 + z_2) + z_1 z_2$$

Compare with $z^2 + \alpha z + \beta = 0$,

coefficient of the z term is $\alpha = -(z_1 + z_2)$.

If $z_1 = a + ib$, then $z_2 = -a + ib$

$$\alpha = -(z_1 + z_2) = -i2b$$

Hence, α need not be real.

It is **not necessary** for α to be real, since z_1 and z_2 need not be complex conjugates.

(d) Given $|z_1| = \frac{\sqrt{26}}{2}$, and $z_1 = (a, b)$ on the ellipse:

$$a^2 + b^2 = \frac{26}{4} = \frac{13}{2}, \quad \frac{a^2}{4} + \frac{b^2}{9} = 1$$

Solving these simultaneously: From the second equation $9a^2 + 4b^2 = 36$. Combined with $a^2 + b^2 = \frac{13}{2}$ (multiply by 4: $4a^2 + 4b^2 = 26$):

$$5a^2 = 10 \implies a = \sqrt{2}, \quad b = \frac{3\sqrt{2}}{2}$$

(Taking $a, b > 0$ since z_1 is in the first quadrant.)

With α not real, take $z_2 = (-a, b) = \left(-\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$:

$$\alpha = -(z_1 + z_2) = -(0 + 3\sqrt{2}i) = -3\sqrt{2}i$$

$$\beta = z_1 z_2 = \left(\sqrt{2} + \frac{3\sqrt{2}}{2}i\right)\left(-\sqrt{2} + \frac{3\sqrt{2}}{2}i\right) = 2\left(1 + \frac{3}{2}i\right)\left(-1 + \frac{3}{2}i\right) = 2\left(-1 - \frac{9}{4}\right) = -\frac{13}{2}$$

Q21 RI P1 Q8

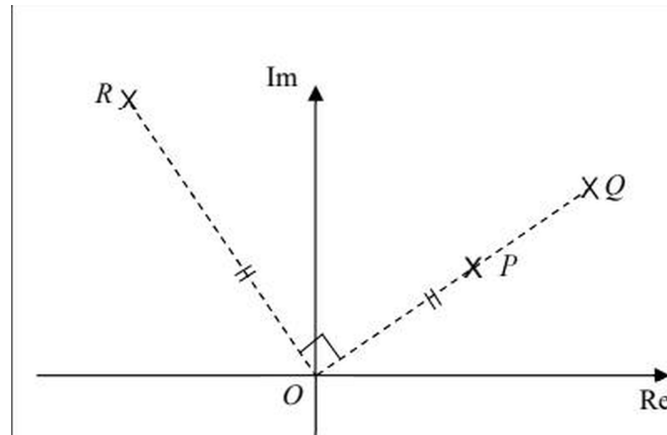
••• [12 marks]

(a)(i) Since $w = u + iv$ with $u > 0$ and $v > 0$, we have $kw = ku + ikv$ with $ku > 0$ and $kv > 0$.

Therefore:

$$\arg(w) = \tan^{-1} \frac{v}{u}, \quad \arg(kw) = \tan^{-1} \frac{kv}{ku} = \tan^{-1} \frac{v}{u} = \arg(w).$$

(a)(ii) Argand diagram showing P , Q and R :

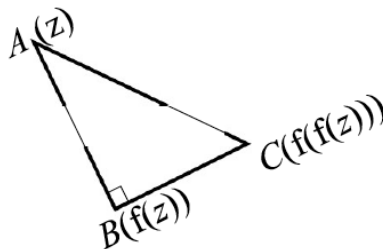


Since kw is w scaled by $k > 1$, Q lies on the ray from O through P , further from O than P (i.e. $OQ > OP$). Since R represents ikw , R is Q rotated 90° anticlockwise about the origin, so $OR = OQ$ and $OR \perp OQ$.

(b)(i)

$$\begin{aligned} f(f(z)) - f(z) &= (z^2 - 2z)^2 - 2(z^2 - 2z) - (z^2 - 2z) \\ &= (z^2 - 2z)[(z^2 - 2z) - 2 - 1] \\ &= (z^2 - 2z)(z^2 - 2z - 3) \\ &= z(z - 2)(z - 3)(z + 1). \end{aligned}$$

(b)(ii) The triangle ABC has a right angle at B with $BC = mBA$ and is described anticlockwise. Rotating \overrightarrow{BC} anticlockwise by $\frac{\pi}{2}$ about B gives \overrightarrow{mBA} :



$$i[f(f(z)) - f(z)] = m(z - f(z))$$

$$\frac{f(f(z)) - f(z)}{f(z) - z} = \frac{m}{i} = -\frac{m}{i} \cdot \frac{i}{i} \cdot \frac{1}{1}$$

$$\frac{z(z - 2)(z - 3)(z + 1)}{z(z - 3)} = mi$$

$$(z - 2)(z + 1) = mi.$$

(b)(iii) With $z = x + 2i$ ($x > 0$):

$$(z - 2)(z + 1) = mi$$

$$z^2 - z - 2 = mi$$

$$(x + 2i)^2 - (x + 2i) - 2 = mi$$

$$x^2 - 4 + 4xi - x - 2i - 2 = mi$$

$$(x^2 - x - 6) + i(4x - 2) = mi$$

Comparing real and imaginary parts:

$$x^2 - x - 6 = 0 \quad \dots (1), \quad 4x - 2 = m \quad \dots (2)$$

From (1): $(x - 3)(x + 2) = 0 \Rightarrow x = 3$ or -2 . Since $x > 0$, $x = 3$.

From (2): $m = 4(3) - 2 = 10$.

B represents $f(z) = z^2 - 2z = (3 + 2i)^2 - 2(3 + 2i) = 9 + 12i - 4 - 6 - 4i = -1 + 8i$.